Thomas Calculus Early Transcendentals 14th Edition Hass SOLUTIONS MANUAL

CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{28-9}{1} = 19$
   (b) $\frac{\Delta f}{\Delta x} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$

2. (a) $\frac{\Delta g}{\Delta x} = \frac{g(3)-g(1)}{3-1} = \frac{3-(-1)}{2} = 2$
   (b) $\frac{\Delta g}{\Delta x} = \frac{g(4)-g(-2)}{4-(-2)} = \frac{8-8}{6} = 0$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right)-h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}-\frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$
   (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right)-h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2}-\frac{\pi}{6}} = \frac{0-\sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(0)}{\pi-0} = \frac{(2-1)-(2+1)}{\pi-0} = -\frac{2}{\pi}$
   (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(-\pi)}{\pi-(-\pi)} = \frac{(2-1)-(2-1)}{2\pi} = 0$

5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2)-R(0)}{2-0} = \frac{\sqrt{8+1}-\sqrt{1}}{2} = \frac{3-1}{2} = 1$

6. $\frac{\Delta P}{\Delta \theta} = \frac{P(2)-P(1)}{2-1} = \frac{(8-16+10)-(1-4+5)}{1} = 2-2 = 0$

7. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-5)-(2^2-5)}{h} = \frac{4+4h+h^2-5+1}{h} = \frac{4h+h^2}{h} = 4+h.$ As $h \to 0,\ 4+h \to 4 \Rightarrow$ at $P(2,-1)$ the slope is 4.
   (b) $y-(-1) = 4(x-2) \Rightarrow y+1 = 4x-8 \Rightarrow y = 4x-9$

8. (a) $\frac{\Delta y}{\Delta x} = \frac{(7-(2+h)^2)-(7-2^2)}{h} = \frac{7-4-4h-h^2-3}{h} = \frac{-4h-h^2}{h} = -4-h.$ As $h \to 0,\ -4-h \to -4 \Rightarrow$ at $P(2,3)$ the slope is -4.
   (b) $y-3 = (-4)(x-2) \Rightarrow y-3 = -4x+8 \Rightarrow y = -4x+11$

9. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-2(2+h)-3)-(2^2-2(2)-3)}{h} = \frac{4+4h+h^2-4-2h-3-(-3)}{h} = \frac{2h+h^2}{h} = 2+h.$ As $h \to 0,\ 2+h \to 2 \Rightarrow$ at $P(2,-3)$ the slope is 2.
   (b) $y-(-3) = 2(x-2) \Rightarrow y+3 = 2x-4 \Rightarrow y = 2x-7.$

10. (a) $\frac{\Delta y}{\Delta x} = \frac{((1+h)^2-4(1+h))-(1^2-4(1))}{h} = \frac{1+2h+h^2-4-4h-(-3)}{h} = \frac{h^2-2h}{h} = h-2.$ As $h \to 0,\ h-2 \to -2 \Rightarrow$ at $P(1,-3)$ the slope is -2.
    (b) $y-(-3) = (-2)(x-1) \Rightarrow y+3 = -2x+2 \Rightarrow y = -2x-1.$

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-2^3}{h} = \frac{8+12h+4h^2+h^3-8}{h} = \frac{12h+4h^2+h^3}{h} = 12+4h+h^2$. As $h \to 0$, $12+4h+h^2 \to 12$, $\Rightarrow$ at $P(2, 8)$ the slope is 12.

(b) $y-8 = 12(x-2) \Rightarrow y-8 = 12x-24 \Rightarrow y = 12x-16$.

12. (a) $\frac{\Delta y}{\Delta x} = \frac{2-(1+h)^3-(2-1^3)}{h} = \frac{2-1-3h-3h^2-h^3-1}{h} = \frac{-3h-3h^2-h^3}{h} = -3-3h-h^2$. As $h \to 0$, $-3-3h-h^2 \to -3$, $\Rightarrow$ at $P(1, 1)$ the slope is -3.

(b) $y-1 = (-3)(x-1) \Rightarrow y-1 = -3x+3 \Rightarrow y = -3x+4$.

13. (a) $\frac{\Delta y}{\Delta x} = \frac{(1+h)^3-12(1+h)-(1^3-12(1))}{h} = \frac{1+3h+3h^2+h^3-12-12h-(-11)}{h} = \frac{-9h+3h^2+h^3}{h} = -9+3h+h^2.$
As $h \to 0$, $-9+3h+h^2 \to -9 \Rightarrow$ at $P(1,-11)$ the slope is -9.
(b) $y-(-11) = (-9)(x-1) \Rightarrow y+11 = -9x+9 \Rightarrow y = -9x-2.$

14. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-3(2+h)^2+4-(2^3-3(2)^2+4)}{h} = \frac{8+12h+6h^2+h^3-12-12h-3h^2+4-0}{h} = \frac{3h^2+h^3}{h} = 3h+h^2.$
As $h \to 0$, $3h+h^2 \to 0 \Rightarrow$ at $P(2,0)$ the slope is 0.
(b) $y-0 = 0(x-2) \Rightarrow y = 0.$

15. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{1}{-2+h}-\frac{1}{-2}}{h} = \frac{2+(-2+h)}{2(-2+h)} \cdot \frac{1}{h} = \frac{1}{2(-2+h)}.$
As $h \to 0$, $\frac{1}{2(-2+h)} \to \frac{-1}{4}$, $\Rightarrow$ at $P\left(-2, \frac{-1}{2}\right)$ the slope is $\frac{-1}{4}$.
(b) $y-\left(\frac{-1}{2}\right) = \frac{-1}{4}(x-(-2)) \Rightarrow y+\frac{1}{2} = \frac{-1}{4}x-\frac{1}{2} \Rightarrow y = \frac{-1}{4}x-1$

16. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{(4+h)}{2-(4+h)}-\frac{4}{2-4}}{h} = \left(\frac{4+h}{-2-h}+\frac{2}{1}\right) \cdot \frac{1}{h} = \frac{4+h+2(-2-h)}{-2-h} \cdot \frac{1}{h} = \frac{-1}{-2-h} = \frac{1}{2+h}.$
As $h \to 0$, $\frac{1}{2+h} \to \frac{1}{2}$, $\Rightarrow$ at $P(4,-2)$ the slope is $\frac{1}{2}$.
(b) $y-(-2) = \frac{1}{2}(x-4) \Rightarrow y+2 = \frac{1}{2}x-2 \Rightarrow y = \frac{1}{2}x-4$

17. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{4+h}-\sqrt{4}}{h} = \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$
As $h \to 0$, $\frac{1}{\sqrt{4+h}+2} \to \frac{1}{\sqrt{4}+2} = \frac{1}{4}$, $\Rightarrow$ at $P(4,2)$ the slope is $\frac{1}{4}$.
(b) $y-2 = \frac{1}{4}(x-4) \Rightarrow y-2 = \frac{1}{4}x-1 \Rightarrow y = \frac{1}{4}x+1$

18. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{7-(-2+h)}-\sqrt{7-(-2)}}{h} = \frac{\sqrt{9-h}-3}{h} = \frac{\sqrt{9-h}-3}{h} \cdot \frac{\sqrt{9-h}+3}{\sqrt{9-h}+3} = \frac{(9-h)-9}{h(\sqrt{9-h}+3)} = \frac{-1}{\sqrt{9-h}+3}.$
As $h \to 0$, $\frac{-1}{\sqrt{9-h}+3} \to \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$, $\Rightarrow$ at $P(-2,3)$ the slope is $\frac{-1}{6}$.
(b) $y-3 = \frac{-1}{6}(x-(-2)) \Rightarrow y-3 = \frac{-1}{6}x-\frac{1}{3} \Rightarrow y = \frac{-1}{6}x+\frac{8}{3}$

19. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(10,225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
$Q_2(14,375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
$Q_3(16.5,475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
$Q_4(18,550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

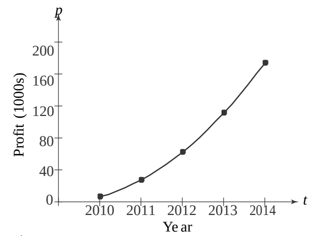
(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

20. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(5,20)$	$\frac{80-20}{10-5} = 12$ m/sec
$Q_2(7,39)$	$\frac{80-39}{10-7} = 13.7$ m/sec
$Q_3(8.5,58)$	$\frac{80-58}{10-8.5} = 14.7$ m/sec
$Q_4(9.5,72)$	$\frac{80-72}{10-9.5} = 16$ m/sec

(b) Approximately 16 m/sec

21. (a)



(b) $\frac{\Delta p}{\Delta t} = \frac{174-62}{2014-2012} = \frac{112}{2} = 56$ thousand dollars per year

(c) The average rate of change from 2011 to 2012 is $\frac{\Delta p}{\Delta t} = \frac{62-27}{2012-2011} = 35$ thousand dollars per year.

The average rate of change from 2012 to 2013 is $\frac{\Delta p}{\Delta t} = \frac{111-62}{2013-2012} = 49$ thousand dollars per year.

So, the rate at which profits were changing in 2012 is approximately $\frac{1}{2}(35+49) = 42$ thousand dollars per year.

22. (a) $F(x) = (x+2)/(x-2)$

x	1.2	1.1	1.01	1.001	1.0001	1
$F(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{0004}$	-3

$\frac{\Delta F}{\Delta x} = \frac{-4.0-(-3)}{1.2-1} = -5.0;$

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{4}-(-3)}{1.1-1} = -4.\overline{4};$

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{04}-(-3)}{1.01-1} = -4.\overline{04};$

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{004}-(-3)}{1.001-1} = -4.\overline{004};$

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{0004}-(-3)}{1.0001-1} = -4.\overline{0004};$

(b) The rate of change of $F(x)$ at $x = 1$ is -4.

23. (a) $\frac{\Delta g}{\Delta x} = \frac{g(2)-g(1)}{2-1} = \frac{\sqrt{2}-1}{2-1} \approx 0.414213$

$\frac{\Delta g}{\Delta x} = \frac{g(1.5)-g(1)}{1.5-1} = \frac{\sqrt{1.5}-1}{0.5} \approx 0.449489$

$\frac{\Delta g}{\Delta x} = \frac{g(1+h)-g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$

(b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$\left(\sqrt{1+h}-1\right)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of $g(x)$ at $x = 1$ is 0.5.

(d) The calculator gives $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

24. (a) i) $\frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{\frac{-1}{6}}{1} = -\frac{1}{6}$

ii) $\frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2}{2T}-\frac{T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = \frac{2-T}{-2T(2-T)} = -\frac{1}{2T},\ T \neq 2$

(b)

T	2.1	2.01	2.001	2.0001	2.00001	2.000001
$f(T)$	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
$(f(T)-f(2))/(T-2)$	-0.2381	-0.2488	-0.2500	-0.2500	-0.2500	-0.2500

(c) The table indicates the rate of change is -0.25 at $t = 2$.

(d) $\lim_{T\to 2}\left(\frac{1}{-2T}\right) = -\frac{1}{4}$

NOTE: Answers will vary in Exercises 25 and 26.

25. (a) $[0, 1]$: $\frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$ mph; $[1, 2.5]$: $\frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$ mph; $[2.5, 3.5]$: $\frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$ mph

(b) At $P\left(\frac{1}{2}, 7.5\right)$: Since the portion of the graph from $t = 0$ to $t = 1$ is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr. At $P(2, 20)$: Since the portion of the graph from $t = 2$ to $t = 2.5$ is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at $t = 2$ is 0 mi/hr.
At $P(3, 22)$:

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3$ is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at $t = 3.5$. Thus for $P(3.5, 30)$

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3.5$ is about 20 mi/hr.

26. (a) $[0, 3]$: $\frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67 \frac{\text{gal}}{\text{day}}$; $[0, 5]$: $\frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2 \frac{\text{gal}}{\text{day}}$; $[7, 10]$: $\frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5 \frac{\text{gal}}{\text{day}}$

(b) At $P(1, 14)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is about -1.45 gal/day.
At $P(4, 6)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -3 gal/day.

(*solution continues on next page*)

At $P(8, 1)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -0.55 gal/day.

(c) It appears that the curve (the consumption) is decreasing the fastest at $t = 3.5$. Thus for $P(3.5, 7.8)$

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at $t = 3.5$ is about -4 gal/day.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

1. (a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$.
   (b) 1 (c) 0 (d) 0.5

2. (a) 0
   (b) -1
   (c) Does not exist. As t approaches 0 from the left, $f(t)$ approaches -1. As t approaches 0 from the right, $f(t)$ approaches 1. There is no single number L that $f(t)$ gets arbitrarily close to as $t \to 0$.
   (d) -1

3. (a) True (b) True (c) False
   (d) False (e) False (f) True
   (g) True (h) False (i) True
   (j) True (k) False

4. (a) False (b) False (c) True
   (d) True (e) True (f) True
   (g) False (h) True (i) False

5. $\lim_{x\to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

6. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim_{x\to 1} \frac{1}{x-1}$ does not exist.

7. Nothing can be said about $f(x)$ because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, $f(x)$ must be arbitrarily close to a single real number L when x is

close enough to x_0. That is, the existence of a limit depends on the values of $f(x)$ for x near x_0, not on the definition of $f(x)$ at x_0 itself.

8. Nothing can be said. In order for $\lim_{x\to 0} f(x)$ to exist, $f(x)$ must close to a single value for x near 0 regardless of the value $f(0)$ itself.

9. No, the definition does not require that f be defined at $x=1$ in order for a limiting value to exist there. If $f(1)$ is defined, it can be any real number, so we can conclude nothing about $f(1)$ from $\lim_{x\to 1} f(x)=5$.

10. No, because the existence of a limit depends on the values of $f(x)$ when x is near 1, not on $f(1)$ itself. If $\lim_{x\to 1} f(x)$ exists, its value may be some number other than $f(1)=5$. We can conclude nothing about $\lim_{x\to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of $f(1)$ alone.

11. $\lim_{x\to -3}(x^2-13)=(-3)^2-13=9-13=-4$

12. $\lim_{x\to 2}(-x^2+5x-2)=-(2)^2+5(2)-2=-4+10-2=4$

13. $\lim_{t\to 6} 8(t-5)(t-7)=8(6-5)(6-7)=-8$

14. $\lim_{x\to -2}(x^3-2x^2+4x+8)=(-2)^3-2(-2)^2+4(-2)+8=-8-8-8+8=-16$

15. $\lim_{x\to 2}\frac{2x+5}{11-x^3}=\frac{2(2)+5}{11-(2)^3}=\frac{9}{3}=3$

16. $\lim_{t\to 2/3}(8-3s)(2s-1)=\left(8-5\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right)-1\right)=(8-2)\left(\left(\frac{4}{3}\right)-1\right)=(6)\left(\frac{1}{3}\right)=2$

17. $\lim_{x\to -1/2}4x(3x+4)^2=4\left(-\frac{1}{2}\right)\left(3\left(-\frac{1}{2}\right)+4\right)^2=(-2)\left(-\frac{3}{2}+4\right)^2=(-2)\left(\frac{5}{2}\right)^2=-\frac{25}{2}$

18. $\lim_{y\to 2}\frac{y+2}{y^2+5y+6}=\frac{2+2}{(2)^2+5(2)+6}=\frac{4}{4+10+6}=\frac{4}{20}=\frac{1}{5}$

19. $\lim_{y\to -3}(5-y)^{4/3}=[5-(-3)]^{4/3}=(8)^{4/3}=\left((8)^{1/3}\right)^4=2^4=16$

20. $\lim_{z\to 4}\sqrt{z^2-10}=\sqrt{4^2-10}=\sqrt{16-10}=\sqrt{6}$

21. $\lim_{h\to 0}\frac{3}{\sqrt{3h+1}+1}=\frac{3}{\sqrt{3(0)+1}+1}=\frac{3}{\sqrt{1}+1}=\frac{3}{2}$

22. $\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}=\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}\cdot\frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2}=\lim_{h\to 0}\frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5h}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5}{\sqrt{5h+4}+2}=\frac{5}{\sqrt{4}+2}=\frac{5}{4}$

23. $\lim_{x\to 5}\frac{x-5}{x^2-25}=\lim_{x\to 5}\frac{x-5}{(x+5)(x-5)}=\lim_{x\to 5}\frac{1}{x+5}=\frac{1}{5+5}=\frac{1}{10}$

24. $\lim_{x\to -3}\frac{x+3}{x^2+4x+3}=\lim_{x\to -3}\frac{x+3}{(x+3)(x+1)}=\lim_{x\to -3}\frac{1}{x+1}=\frac{1}{-3+1}=-\frac{1}{2}$

25. $\lim\limits_{x\to-5}\frac{x^2+3x-10}{x+5} = \lim\limits_{x\to-5}\frac{(x+5)(x-2)}{x+5} = \lim\limits_{x\to-5}(x-2) = -5-2 = -7$

26. $\lim\limits_{x\to2}\frac{x^2-7x-10}{x-2} = \lim\limits_{x\to2}\frac{(x-5)(x-2)}{x-2} = \lim\limits_{x\to2}(x-5) = 2-5 = -3$

27. $\lim\limits_{t\to1}\frac{t^2+t-2}{t^2-1} = \lim\limits_{t\to1}\frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim\limits_{t\to1}\frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

28. $\lim\limits_{t\to-1}\frac{t^2+3t+2}{t^2-t-2} = \lim\limits_{t\to-1}\frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim\limits_{t\to-1}\frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

29. $\lim\limits_{x\to-2}\frac{-2x-4}{x^3+2x^2} = \lim\limits_{x\to-2}\frac{-2(x+2)}{x^2(x+2)} = \lim\limits_{x\to-2}\frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

30. $\lim\limits_{y\to0}\frac{5y^3+8y^2}{3y^4-16y^2} = \lim\limits_{y\to0}\frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim\limits_{y\to0}\frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$

31. $\lim\limits_{x\to1}\frac{x^{-1}-1}{x-1} = \lim\limits_{x\to1}\frac{\frac{1-x}{x}}{x-1} = \lim\limits_{x\to1}\left(\frac{1-x}{x}\cdot\frac{1}{x-1}\right) = \lim\limits_{x\to1} -\frac{1}{x} = -1$

32. $\lim\limits_{x\to0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim\limits_{x\to0}\frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim\limits_{x\to0}\left(\frac{2x}{(x-1)(x+1)}\cdot\frac{1}{x}\right) = \lim\limits_{x\to0}\frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$

33. $\lim\limits_{u\to1}\frac{u^4-1}{u^3-1} = \lim\limits_{u\to1}\frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim\limits_{u\to1}\frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

34. $\lim\limits_{v\to2}\frac{v^3-8}{v^4-16} = \lim\limits_{v\to2}\frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim\limits_{v\to2}\frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$

35. $\lim\limits_{x\to9}\frac{\sqrt{x}-3}{x-9} = \lim\limits_{x\to9}\frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim\limits_{x\to9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

36. $\lim\limits_{x\to4}\frac{4x-x^2}{2-\sqrt{x}} = \lim\limits_{x\to4}\frac{x(4-x)}{2-\sqrt{x}} = \lim\limits_{x\to4}\frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim\limits_{x\to4} x\left(2+\sqrt{x}\right) = 4(2+2) = 16$

37. $\lim\limits_{x\to1}\frac{x-1}{\sqrt{x+3}-2} = \lim\limits_{x\to1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)} = \lim\limits_{x\to1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{(x+3)-4} = \lim\limits_{x\to1}\left(\sqrt{x+3}+2\right) = \sqrt{4}+2 = 4$

38. $\lim\limits_{x\to-1}\frac{\sqrt{x^2+8}-3}{x+1} = \lim\limits_{x\to-1}\frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to-1}\frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim\limits_{x\to-1}\frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim\limits_{x\to-1}\frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

39. $\lim\limits_{x\to2}\frac{\sqrt{x^2+12}-4}{x-2} = \lim\limits_{x\to2}\frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim\limits_{x\to2}\frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim\limits_{x\to2}\frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$

$= \lim\limits_{x\to2}\frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

$x \to 2$ $\sqrt{x^2+12}+4$ 16+4 2

40. $\lim\limits_{x\to -2}\frac{x+2}{\sqrt{x^2+5}-3}=\lim\limits_{x\to -2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{\left(\sqrt{x^2+5}-3\right)\left(\sqrt{x^2+5}+3\right)}=\lim\limits_{x\to -2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x^2+5)-9}=\lim\limits_{x\to -2}\frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x+2)(x-2)}$
$=\lim\limits_{x\to -2}\frac{\sqrt{x^2+5}+3}{x-2}=\frac{\sqrt{9}+3}{-4}=-\frac{3}{2}$

41. $\lim\limits_{x\to -3}\frac{2-\sqrt{x^2-5}}{x+3}=\lim\limits_{x\to -3}\frac{\left(2-\sqrt{x^2-5}\right)\left(2+\sqrt{x^2-5}\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to -3}\frac{4-(x^2-5)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to -3}\frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$
$=\lim\limits_{x\to -3}\frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)}=\lim\limits_{x\to -3}\frac{3-x}{2+\sqrt{x^2-5}}=\frac{6}{2+\sqrt{4}}=\frac{3}{2}$

42. $\lim\limits_{x\to 4}\frac{4-x}{5-\sqrt{x^2+9}}=\lim\limits_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{\left(5-\sqrt{x^2+9}\right)\left(5+\sqrt{x^2+9}\right)}=\lim\limits_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-(x^2+9)}=\lim\limits_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2}$
$=\lim\limits_{x\to 4}\frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)}=\lim\limits_{x\to 4}\frac{5+\sqrt{x^2+9}}{4+x}=\frac{5+\sqrt{25}}{8}=\frac{5}{4}$

43. $\lim\limits_{x\to 0}(2\sin x-1)=2\sin 0-1=0-1=-1$

44. $\lim\limits_{x\to 0}\sin^2 x=\left(\lim\limits_{x\to 0}\sin x\right)^2=(\sin 0)^2=0^2=0$

45. $\lim\limits_{x\to 0}\sec x=\lim\limits_{x\to 0}\frac{1}{\cos x}=\frac{1}{\cos 0}=\frac{1}{1}=1$

46. $\lim\limits_{x\to 0}\tan x=\lim\limits_{x\to 0}\frac{\sin x}{\cos x}=\frac{\sin 0}{\cos 0}=\frac{0}{1}=0$

47. $\lim\limits_{x\to 0}\frac{1+x+\sin x}{3\cos x}=\frac{1+0+\sin 0}{3\cos 0}=\frac{1+0+0}{3}=\frac{1}{3}$

48. $\lim\limits_{x\to 0}(x^2-1)(2-\cos x)=(0^2-1)(2-\cos 0)=(-1)(2-1)=(-1)(1)=-1$

49. $\lim\limits_{x\to -\pi}\sqrt{x+4}\cos(x+\pi)=\lim\limits_{x\to -\pi}\sqrt{x+4}\cdot\lim\limits_{x\to -\pi}\cos(x+\pi)=\sqrt{-\pi+4}\cdot\cos 0=\sqrt{4-\pi}\cdot 1=\sqrt{4-\pi}$

50. $\lim\limits_{x\to 0}\sqrt{7+\sec^2 x}=\sqrt{\lim\limits_{x\to 0}(7+\sec^2 x)}=\sqrt{7+\lim\limits_{x\to 0}\sec^2 x}=\sqrt{7+\sec^2 0}=\sqrt{7+(1)^2}=2\sqrt{2}$

51. (a) quotient rule
(b) difference and power rules
(c) sum and constant multiple rules

52. (a) quotient rule
(b) power and product rules
(c) difference and constant multiple rules

53. (a) $\lim\limits_{x\to c}f(x)g(x)=\left[\lim\limits_{x\to c}f(x)\right]\left[\lim\limits_{x\to c}g(x)\right]=(5)(-2)=-10$
(b) $\lim\limits_{x\to c}2f(x)g(x)=2\left[\lim\limits_{x\to c}f(x)\right]\left[\lim\limits_{x\to c}g(x)\right]=2(5)(-2)=-20$
(c) $\lim\limits_{x\to c}[f(x)+3g(x)]=\lim\limits_{x\to c}f(x)+3\lim\limits_{x\to c}g(x)=5+3(-2)=-1$
(d) $\lim\limits_{x\to c}\frac{f(x)}{f(x)-g(x)}=\frac{\lim\limits_{x\to c}f(x)}{\lim\limits_{x\to c}f(x)-\lim\limits_{x\to c}g(x)}=\frac{5}{5-(-2)}=\frac{5}{7}$

54. (a) $\lim_{x\to 4}[g(x)+3]=\lim_{x\to 4} g(x)+\lim_{x\to 4} 3=-3+3=0$

(b) $\lim_{x\to 4} xf(x)=\lim_{x\to 4} x\cdot\lim_{x\to 4} f(x)=(4)(0)=0$

(c) $\lim_{x\to 4}[g(x)]^2=\left[\lim_{x\to 4} g(x)\right]^2=[-3]^2=9$

(d) $\lim_{x\to 4}\frac{g(x)}{f(x)-1}=\frac{\lim_{x\to 4} g(x)}{\lim_{x\to 4} f(x)-\lim_{x\to 4} 1}=\frac{-3}{0-1}=3$

55. (a) $\lim_{x\to b}[f(x)+g(x)]=\lim_{x\to b} f(x)+\lim_{x\to b} g(x)=7+(-3)=4$

(b) $\lim_{x\to b} f(x)\cdot g(x)=\left[\lim_{x\to b} f(x)\right]\left[\lim_{x\to b} g(x)\right]=(7)(-3)=-21$

(c) $\lim_{x\to b} 4g(x)=\left[\lim_{x\to b} 4\right]\left[\lim_{x\to b} g(x)\right]=(4)(-3)=-12$

(d) $\lim_{x\to b} f(x)/g(x)=\lim_{x\to b} f(x)/\lim_{x\to b} g(x)=\frac{7}{-3}=-\frac{7}{3}$

56. (a) $\lim_{x\to -2}[p(x)+r(x)+s(x)]=\lim_{x\to -2} p(x)+\lim_{x\to -2} r(x)+\lim_{x\to -2} s(x)=4+0+(-3)=1$

(b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)=\left[\lim_{x\to -2} p(x)\right]\left[\lim_{x\to -2} r(x)\right]\left[\lim_{x\to -2} s(x)\right]=(4)(0)(-3)=0$

(c) $\lim_{x\to -2}[-4p(x)+5r(x)]/s(x)=\left[-4\lim_{x\to -2} p(x)+5\lim_{x\to -2} r(x)\right]\Big/\lim_{x\to -2} s(x)=[-4(4)+5(0)]/-3=\frac{16}{3}$

57. $\lim_{h\to 0}\frac{(1+h)^2-1^2}{h}=\lim_{h\to 0}\frac{1+2h+h^2-1}{h}=\lim_{h\to 0}\frac{h(2+h)}{h}=\lim_{h\to 0}(2+h)=2$

58. $\lim_{h\to 0}\frac{(-2+h)^2-(-2)^2}{h}=\lim_{h\to 0}\frac{4-4h+h^2-4}{h}=\lim_{h\to 0}\frac{h(h-4)}{h}=\lim_{h\to 0}(h-4)=-4$

59. $\lim_{h\to 0}\frac{[3(2+h)-4]-[3(2)-4]}{h}=\lim_{h\to 0}\frac{3h}{h}=3$

60. $\lim_{h\to 0}\frac{\left(\frac{1}{-2+h}\right)-\left(\frac{1}{-2}\right)}{h}=\lim_{h\to 0}\frac{\frac{-2}{-2+h}-1}{-2h}=\lim_{h\to 0}\frac{-2-(-2+h)}{-2h(-2+h)}=\lim_{h\to 0}\frac{-h}{h(4-2h)}=-\frac{1}{4}$

61. $\lim_{h\to 0}\frac{\sqrt{7+h}-\sqrt{7}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)}=\lim_{h\to 0}\frac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{7+h}+\sqrt{7}}=\frac{1}{2\sqrt{7}}$

62. $\lim_{h\to 0}\frac{\sqrt{3(0+h)+1}-\sqrt{3(0)+1}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{3h+1}-1\right)\left(\sqrt{3h+1}+1\right)}{h\left(\sqrt{3h+1}+1\right)}=\lim_{h\to 0}\frac{(3h+1)-1}{h\left(\sqrt{3h+1}+1\right)}=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3h+1}+1\right)}=\lim_{h\to 0}\frac{3}{\sqrt{3h+1}+1}=\frac{3}{2}$

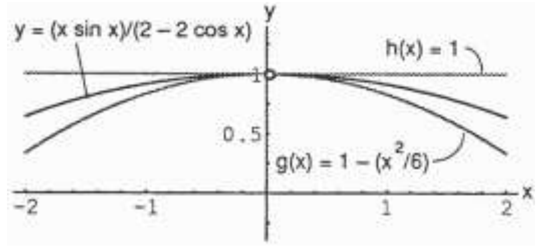
63. $\lim_{x\to 0}\sqrt{5-2x^2}=\sqrt{5-2(0)^2}=\sqrt{5}$ and $\lim_{x\to 0}\sqrt{5-x^2}=\sqrt{5-(0)^2}=\sqrt{5}$; by the sandwich theorem, $\lim_{x\to 0} f(x)=\sqrt{5}$

64. $\lim_{x\to 0}(2-x^2)=2-0=2$ and $\lim_{x\to 0} 2\cos x=2(1)=2$; by the sandwich theorem, $\lim_{x\to 0} g(x)=2$

65. (a) $\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)=1-\frac{0}{6}=1$ and $\lim_{x\to 0} 1=1$; by the sandwich theorem, $\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}=1$

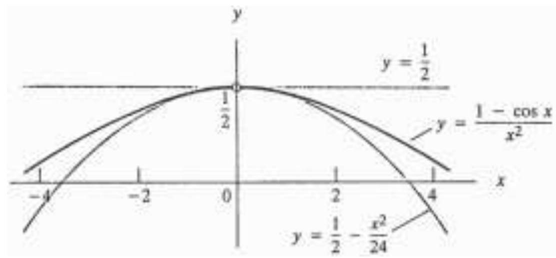
(b)

(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



66. (a) $\lim_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{24}\right) = \lim_{x\to 0}\frac{1}{2} - \lim_{x\to 0}\frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$; by the sandwich theorem, $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1-\cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



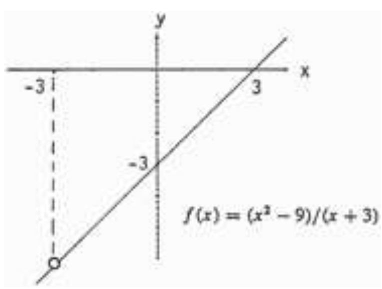
67. (a) $f(x) = (x^2 - 9)/(x+3)$

x	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

x	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

The estimate is $\lim_{x\to -3} f(x) = -6$.

(b)

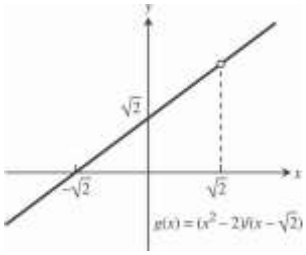


(c) $f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim_{x\to -3}(x-3) = -3-3 = -6$.

68. (a) $g(x) = (x^2 - 2)/\left(x - \sqrt{2}\right)$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
$g(x)$	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



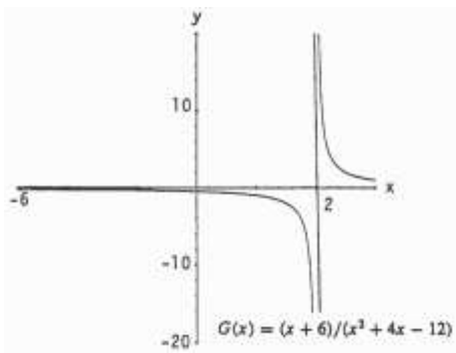
(c) $g(x) = \frac{x^2-2}{x-\sqrt{2}} = \frac{(x+\sqrt{2})(x-\sqrt{2})}{(x-\sqrt{2})} = x+\sqrt{2}$ if $x \neq \sqrt{2}$, and $\lim_{x\to\sqrt{2}} \left(x+\sqrt{2}\right) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$.

69. (a) $G(x) = (x+6)/(x^2+4x-12)$

x	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999
$G(x)$	$-.126582$	$-.1251564$	$-.1250156$	$-.1250015$	$-.1250001$	$-.1250000$

x	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001
$G(x)$	$-.123456$	$-.124843$	$-.124984$	$-.124998$	$-.124999$	$-.124999$

(b)



(c) $G(x) = \frac{x+6}{(x^2+4x-12)} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2}$ if $x \neq -6$, and $\lim_{x\to-6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125$.

70. (a) $h(x) = (x^2-2x-3)/(x^2-4x+3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
$h(x)$	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$h(x)$	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)

(c) $h(x) = \frac{x^2-2x-3}{x^2-4x+3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1}$ if $x \neq 3$, and $\lim \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2$

3 $x-1$ = (b) $3-1$ = 2 =

$x \to$

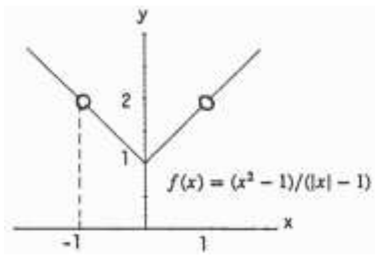
(b)

71. (a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



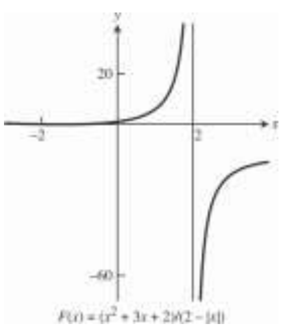
(c) $f(x) = \frac{x^2-1}{|x|-1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \ge 0 \text{ and } x \ne 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \ne -1 \end{cases}$, and $\lim\limits_{x \to -1} (1-x) = 1-(-1) = 2.$

72. (a) $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
$F(x)$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

x	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
$F(x)$	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$

(b)



(c) $F(x) = \frac{x^2+3x+2}{2-|x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \ge 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \ne -2 \end{cases}$, and $\lim\limits_{x \to -2} (x+1) = -2+1 = -1.$

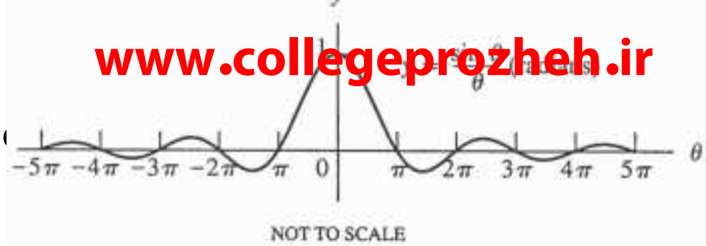
73. (a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim\limits_{\theta \to 0} g(\theta) = 1$

(b)



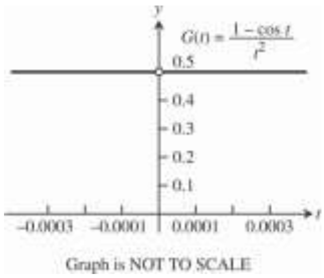
74. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

t	−.1	−.01	−.001	−.0001	−.00001	−.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$\lim_{t \to 0} G(t) = 0.5$

(b)



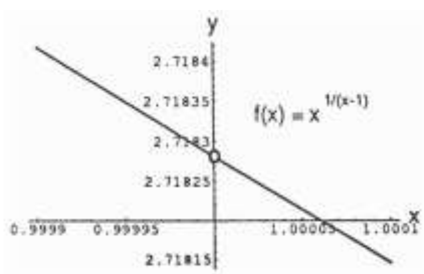
75. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$\lim_{x \to 1} f(x) \approx 0.36788$

(b)



Graph is NOT TO SCALE. Also, the intersection of the axes is not the origin: the axes intersect at the point (1, 2.71820).

(b)

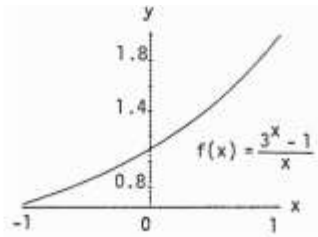
76. (a) $f(x) = (3^x - 1)/x$

x	.1	.01	.001	.0001	.00001	.000001
$f(x)$	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

x	−.1	−.01	−.001	−.0001	−.00001	−.000001
$f(x)$	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$\lim\limits_{x\to 1} f(x) \approx 1.0986$

(b)



77. $\lim\limits_{x\to c} f(x)$ exists at those points c where $\lim\limits_{x\to c} x^4 = \lim\limits_{x\to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0 \Rightarrow c = 0, 1,$ or -1. Moreover, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to 1} f(x) = 1$.

78. Nothing can be concluded about the values of f, g, and h at $x = 2$. Yes, $f(2)$ could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim\limits_{x\to 2} f(x) = -5 \neq 0$.

79. $1 = \lim\limits_{x\to 4} \frac{f(x)-5}{x-2} = \frac{\lim\limits_{x\to 4} f(x) - \lim\limits_{x\to 4} 5}{\lim\limits_{x\to 4} x - \lim\limits_{x\to 4} 2} = \frac{\lim\limits_{x\to 4} f(x) - 5}{4-2} \Rightarrow \lim\limits_{x\to 4} f(x) - 5 = 2(1) \Rightarrow \lim\limits_{x\to 4} f(x) = 2 + 5 = 7.$

80. (a) $1 = \lim\limits_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim\limits_{x\to -2} f(x)}{\lim\limits_{x\to -2} x^2} = \frac{\lim\limits_{x\to -2} f(x)}{4} \Rightarrow \lim\limits_{x\to -2} f(x) = 4.$

(b) $1 = \lim\limits_{x\to -2} \frac{f(x)}{x^2} = \left[\lim\limits_{x\to -2} \frac{f(x)}{x}\right]\left[\lim\limits_{x\to -2} \frac{1}{x}\right] = \left[\lim\limits_{x\to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim\limits_{x\to -2} \frac{f(x)}{x} = -2.$

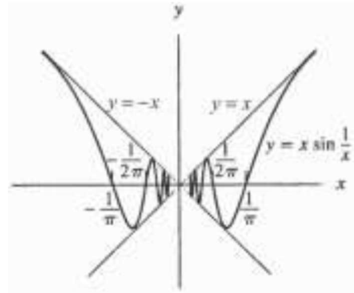
81. (a) $0 = 3\cdot 0 = \left[\lim\limits_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim\limits_{x\to 2} (x-2)\right] = \lim\limits_{x\to 2}\left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim\limits_{x\to 2} [f(x)-5]$
$= \lim\limits_{x\to 2} f(x) - 5 \Rightarrow \lim\limits_{x\to 2} f(x) = 5.$

(b) $0 = 4\cdot 0 = \left[\lim\limits_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim\limits_{x\to 2} (x-2)\right] \Rightarrow \lim\limits_{x\to 2} f(x) = 5$ as in part (a).

82. (a) $0 = 1\cdot 0 = \left[\lim\limits_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim\limits_{x\to 0} x\right]^2 = \left[\lim\limits_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim\limits_{x\to 0} x^2\right] = \lim\limits_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x^2\right] = \lim\limits_{x\to 0} f(x).$
That is, $\lim\limits_{x\to 0} f(x) = 0.$

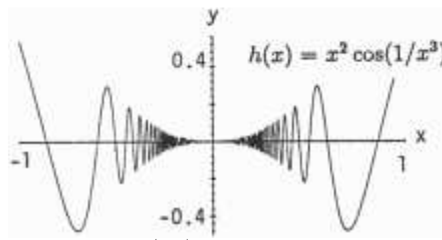
(b) $0 = 1\cdot 0 = \left[\lim\limits_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim\limits_{x\to 0} x\right] = \lim\limits_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x\right] = \lim\limits_{x\to 0} \frac{f(x)}{x}$. That is, $\lim\limits_{x\to 0} \frac{f(x)}{x} = 0.$

83. (a) $\lim_{x\to0} x\sin\frac{1}{x}=0$



(b) $-1\le\sin\frac{1}{x}\le1$ for $x\ne0$:
$x>0\Rightarrow -x\le x\sin\frac{1}{x}\le x\Rightarrow \lim_{x\to0} x\sin\frac{1}{x}=0$ by the sandwich theorem;
$x<0\Rightarrow -x\ge x\sin\frac{1}{x}\ge x\Rightarrow \lim_{x\to0} x\sin\frac{1}{x}=0$ by the sandwich theorem.

84. (a) $\lim_{x\to0} x^2\cos\left(\frac{1}{x^3}\right)=0$



(b) $-1\le\cos\left(\frac{1}{x^3}\right)\le1$ for $x\ne0\Rightarrow -x^2\le x^2\cos\left(\frac{1}{x^3}\right)\le x^2\Rightarrow \lim_{x\to0} x^2\cos\left(\frac{1}{x^3}\right)=0$ by the sandwich theorem since $\lim_{x\to0} x^2=0$.

85–90. Example CAS commands:

Maple:

```
f := x -> (x^4 − 16)/(x − 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
        title = "Section 2.2, #85(a)" );
limit( f(x), x = x0 );
```

In Exercise 87, note that the standard cube root, x^(1/3), is not defined for x<0 in many CASs. This can be overcome in Maple by entering the function as f := x -> (surd(x+1, 3) − 1)/x.

Mathematica: (assigned function and values for x0 and h may vary)

```
Clear[f, x]
f[x_]:=(x^3 − x^2 − 5x − 3)/(x + 1)^2
x0= −1; h = 0.1;
Plot[f[x],{x, x0 − h, x0 + h}]
Limit[f[x], x → x0]
```

2.3 THE PRECISE DEFINITION OF A LIMIT

1. 

Step 1: $|x-5|<\delta\Rightarrow -\delta<x-5<\delta\Rightarrow -\delta+5<x<\delta+5$
Step 2: $\delta+5=7\Rightarrow\delta=2$, or $-\delta+5=1\Rightarrow\delta=4$.

The value of δ which assures $|x-5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

2. 

Step 1: $|x-2|<\delta \Rightarrow -\delta < x-2<\delta \Rightarrow -\delta+2<x<\delta+2$

Step 2: $-\delta+2=1 \Rightarrow \delta=1$, or $\delta+2=7 \Rightarrow \delta=5$.

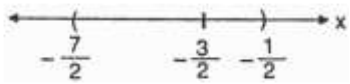
The value of δ which assures $|x-2|<\delta \Rightarrow 1<x<7$ is the smaller value, $\delta=1$.

3. 

Step 1: $|x-(-3)|<\delta \Rightarrow -\delta < x+3<\delta \Rightarrow -\delta-3<x<\delta-3$

Step 2: $-\delta-3=-\frac{7}{2} \Rightarrow \delta=\frac{1}{2}$, or $\delta-3=-\frac{1}{2} \Rightarrow \delta=\frac{5}{2}$.

The value of δ which assures $|x-(-3)|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=\frac{1}{2}$.

4. 

Step 1: $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\delta < x+\frac{3}{2}<\delta \Rightarrow -\delta-\frac{3}{2}<x<\delta-\frac{3}{2}$

Step 2: $-\delta-\frac{3}{2}=-\frac{7}{2} \Rightarrow \delta=2$, or $\delta-\frac{3}{2}=-\frac{1}{2} \Rightarrow \delta=1$.

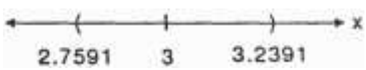
The value of δ which assures $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=1$.

5. 

Step 1: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta < x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$

Step 2: $-\delta+\frac{1}{2}=\frac{4}{9} \Rightarrow \delta=\frac{1}{18}$, or $\delta+\frac{1}{2}=\frac{4}{7} \Rightarrow \delta=\frac{1}{14}$.

The value of δ which assures $\left|x-\frac{1}{2}\right|<\delta \Rightarrow \frac{4}{9}<x<\frac{4}{7}$ is the smaller value, $\delta=\frac{1}{18}$.

6. 

Step 1: $|x-3|<\delta \Rightarrow -\delta < x-3<\delta \Rightarrow -\delta+3<x<\delta+3$

Step 2: $-\delta+3=2.7591 \Rightarrow \delta=0.2409$, or $\delta+3=3.2391 \Rightarrow \delta=0.2391$.

The value of δ which assures $|x-3|<\delta \Rightarrow 2.7591<x<3.2391$ is the smaller value, $\delta=0.2391$.

7. Step 1: $|x-5|<\delta \Rightarrow -\delta < x-5<\delta \Rightarrow -\delta+5<x<\delta+5$

Step 2: From the graph, $-\delta+5=4.9 \Rightarrow \delta=0.1$, or $\delta+5=5.1 \Rightarrow \delta=0.1$; thus $\delta=0.1$ in either case.

8. Step 1: $|x-(-3)|<\delta \Rightarrow -\delta < x+3<\delta \Rightarrow -\delta-3<x<\delta-3$

Step 2: From the graph, $-\delta-3=-3.1 \Rightarrow \delta=0.1$, or $\delta-3=-2.9 \Rightarrow \delta=0.1$; thus $\delta=0.1$.

9. Step 1: $|x-1|<\delta \Rightarrow -\delta < x-1<\delta \Rightarrow -\delta+1<x<\delta+1$

Step 2: From the graph, $-\delta+1=\frac{9}{16} \Rightarrow \delta=\frac{7}{16}$, or $\delta+1=\frac{25}{16} \Rightarrow \delta=\frac{9}{16}$; thus $\delta=\frac{7}{16}$.

10. Step 1: $|x-3|<\delta \Rightarrow -\delta < x-3<\delta \Rightarrow -\delta+3<x<\delta+3$

Step 2: From the graph, $-\delta+3=2.61 \Rightarrow \delta=0.39$, or $\delta+3=3.41 \Rightarrow \delta=0.41$; thus $\delta=0.39$.

11. Step 1: $|x-2|<\delta \Rightarrow -\delta < x-2<\delta \Rightarrow -\delta +2 < x < \delta + 2$
Step 2: From the graph, $-\delta + 2 = \sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$, or $\delta + 2 = \sqrt{5} \Rightarrow \delta = \sqrt{5}-2 \approx 0.2361$; thus $\delta = \sqrt{5}-2$.

12. Step 1: $|x-(-1)|<\delta \Rightarrow -\delta < x+1<\delta \Rightarrow -\delta -1< x < \delta -1$
Step 2: From the graph, $-\delta -1 = -\frac{\sqrt{5}}{2} \Rightarrow \delta = \frac{\sqrt{5}-2}{2} \approx 0.118$ or $\delta -1 = -\frac{\sqrt{3}}{2} \Rightarrow \delta = \frac{2-\sqrt{3}}{2} \approx 0.1340$; thus $\delta = \frac{\sqrt{5}-2}{2}$.

13. Step 1: $|x-(-1)| <\delta \Rightarrow -\delta < x+1<\delta \Rightarrow -\delta -1< x < \delta -1$
Step 2: From the graph, $-\delta -1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$, or $\delta -1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$; thus $\delta = \frac{9}{25} = 0.36$.

14. Step 1: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta < x-\frac{1}{2}<\delta \Rightarrow -\delta +\frac{1}{2}< x < \delta + \frac{1}{2}$
Step 2: From the graph, $-\delta + \frac{1}{2} = \frac{1}{2.01} \Rightarrow \delta = \frac{1}{2} - \frac{1}{2.01} \approx 0.00248$, or $\delta + \frac{1}{2} = \frac{1}{1.99} \Rightarrow \delta = \frac{1}{1.99} - \frac{1}{2} \approx 0.00251$; thus $\delta = 0.00248$.

15. Step 1: $|(x+1)-5|< 0.01 \Rightarrow |x-4|< 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x-4|<\delta \Rightarrow -\delta < x-4<\delta \Rightarrow -\delta +4< x < \delta +4 \Rightarrow \delta = 0.01$.

16. Step 1: $|(2x-2)-(-6)|< 0.02 \Rightarrow |2x+4|< 0.02 \Rightarrow -0.02 < 2x+4 < 0.02$
$\Rightarrow -4.02 < 2x < -3.98 \Rightarrow -2.01 < x < -1.99$
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta < x+2<\delta \Rightarrow -\delta -2< x < \delta -2 \Rightarrow \delta = 0.01$.

17. Step 1: $\left|\sqrt{x+1} -1\right| < 0.1 \Rightarrow -0.1 < \sqrt{x+1} -1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
$\Rightarrow -0.19 < x < 0.21$
Step 2: $|x-0|<\delta \Rightarrow -\delta < x<\delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.

18. Step 1: $\left|\sqrt{x}-\frac{1}{2}\right|< 0.1 \Rightarrow -0.1 < \sqrt{x}-\frac{1}{2} < 0.1 \Rightarrow 0.4 < \sqrt{x} < 0.6 \Rightarrow 0.16 < x < 0.36$
Step 2: $\left|x-\frac{1}{4}\right|<\delta \Rightarrow -\delta < x-\frac{1}{4}<\delta \Rightarrow$
Then $-\delta + \frac{1}{4} = 0.16 \Rightarrow \delta = 0.09$ or $\delta + \frac{1}{4} = 0.36 \Rightarrow \delta = 0.11$; thus $\delta = 0.09$.

19. Step 1: $\left|\sqrt{19-x}-3\right|<1 \Rightarrow -1<\sqrt{19-x}-3<1 \Rightarrow 2<\sqrt{19-x}<4 \Rightarrow 4<19-x<16$
$\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
Step 2: $|x-10|<\delta \Rightarrow -\delta < x-10<\delta \Rightarrow -\delta +10< x < \delta +10$.
Then $-\delta +10 = 3 \Rightarrow \delta = 7$, or $\delta +10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.

20. Step 1: $\left|\sqrt{x-7}-4\right|<1 \Rightarrow -1<\sqrt{x-7}-4<1 \Rightarrow 3<\sqrt{x-7}<5 \Rightarrow 9<x-7<25 \Rightarrow 16<x<32$
Step 2: $|x-23|<\delta \Rightarrow -\delta < x-23<\delta \Rightarrow -\delta +23< x < \delta +23$.
Then $-\delta + 23 = 16 \Rightarrow \delta = 7$, or $\delta + 23 = 32 \Rightarrow \delta = 9$; thus $\delta = 7$.

21. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right|< 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
Step 2: $|x-4|<\delta \Rightarrow -\delta < x-4<\delta \Rightarrow -\delta +4< x < \delta +4$.
Then $-\delta + 4 = \frac{10}{3}$ or $\delta = \frac{2}{3}$, or $\delta + 4 = 5$ or $\delta = 1$; thus $\delta = \frac{2}{3}$.

22. Step 1: $\left|x^2-3\right|<0.1 \Rightarrow -0.1<x^2-3<0.1 \Rightarrow 2.9<x^2<3.1 \Rightarrow \sqrt{2.9}<x<\sqrt{3.1}$

Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow -\delta+\sqrt{3}<x<\delta+\sqrt{3}$.

Then $-\delta+\sqrt{3}=\sqrt{2.9} \Rightarrow \delta=\sqrt{3}-\sqrt{2.9}\approx 0.0291$, or $\delta+\sqrt{3}=\sqrt{3.1} \Rightarrow \delta=\sqrt{3.1}-\sqrt{3}\approx 0.0286$; thus $\delta=0.0286$

23. Step 1: $\left|x^2-4\right|<0.5 \Rightarrow -0.5<x^2-4<0.5 \Rightarrow 3.5<x^2<4.5 \Rightarrow \sqrt{3.5}<|x|<\sqrt{4.5} \Rightarrow -\sqrt{4.5}<x<-\sqrt{3.5}$, for x near -2.

Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.

Then $-\delta-2=-\sqrt{4.5} \Rightarrow \delta=\sqrt{4.5}-2\approx 0.1213$, or $\delta-2=-\sqrt{3.5} \Rightarrow \delta=2-\sqrt{3.5}\approx 0.1292$; thus $\delta=\sqrt{4.5}-2\approx 0.12$.

24. Step 1: $\left|\frac{1}{x}-(-1)\right|<0.1 \Rightarrow -0.1<\frac{1}{x}+1<0.1 \Rightarrow -\frac{11}{10}<\frac{1}{x}<-\frac{9}{10} \Rightarrow -\frac{10}{11}>x>-\frac{10}{9}$ or $-\frac{10}{9}<x<-\frac{10}{11}$

Step 2: $|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta \Rightarrow -\delta-1<x<\delta-1$.

Then $-\delta-1=-\frac{10}{9} \Rightarrow \delta=\frac{1}{9}$, or $\delta-1=-\frac{10}{11} \Rightarrow \delta=\frac{1}{11}$; thus $\delta=\frac{1}{11}$.

25. Step 1: $\left|(x^2-5)-11\right|<1 \Rightarrow \left|x^2-16\right|<1 \Rightarrow -1<x^2-16<1 \Rightarrow 15<x^2<17 \Rightarrow \sqrt{15}<x<\sqrt{17}$.

Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.

Then $-\delta+4=\sqrt{15} \Rightarrow \delta=4-\sqrt{15}\approx 0.1270$, or $\delta+4=\sqrt{17} \Rightarrow \delta=\sqrt{17}-4\approx 0.1231$; thus $\delta=\sqrt{17}-4\approx 0.12$.

26. Step 1: $\left|\frac{120}{x}-5\right|<1 \Rightarrow -1<\frac{120}{x}-5<1 \Rightarrow 4<\frac{120}{x}<6 \Rightarrow \frac{1}{4}>\frac{x}{120}>\frac{1}{6} \Rightarrow 30>x>20$ or $20<x<30$.

Step 2: $|x-24|<\delta \Rightarrow -\delta<x-24<\delta \Rightarrow -\delta+24<x<\delta+24$.

Then $-\delta+24=20 \Rightarrow \delta=4$, or $\delta+24=30 \Rightarrow \delta=6$; thus $\Rightarrow \delta=4$.

27. Step 1: $|mx-2m|<0.03 \Rightarrow -0.03<mx-2m<0.03 \Rightarrow -0.03+2m<mx<0.03+2m \Rightarrow 2-\frac{0.03}{m}<x<2+\frac{0.03}{m}$.

Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.

Then $-\delta+2=2-\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$, or $\delta+2=2+\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$. In either case, $\delta=\frac{0.03}{m}$.

28. Step 1: $|mx-3m|<c \Rightarrow -c<mx-3m<c \Rightarrow -c+3m<mx<c+3m \Rightarrow 3-\frac{c}{m}<x<3+\frac{c}{m}$

Step 2: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.

Then $-\delta+3=3-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+3=3+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

29. Step 1: $\left|(mx+b)-\left(\frac{m}{2}+b\right)\right|<c \Rightarrow -c<mx-\frac{m}{2}<c \Rightarrow -c+\frac{m}{2}<mx<c+\frac{m}{2} \Rightarrow \frac{1}{2}-\frac{c}{m}<x<\frac{1}{2}+\frac{c}{m}$.

Step 2: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta<x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$.

Then $-\delta+\frac{1}{2}=\frac{1}{2}-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+\frac{1}{2}=\frac{1}{2}+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

30. Step 1: $|(mx+b)-(m+b)|<0.05 \Rightarrow -0.05<mx-m<0.05 \Rightarrow -0.05+m<mx<0.05+m$

$\Rightarrow 1-\frac{0.05}{m}<x<1+\frac{0.05}{m}$.

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.

Then $-\delta+1=1-\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$, or $\delta+1=1+\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$. In either case, $\delta=\frac{0.05}{m}$.

31. $\lim\limits_{x\to 3}(3-2x)=3-2(3)=-3$

Step 1: $|(3-2x)-(-3)|<0.02 \Rightarrow -0.02<6-2x<0.02 \Rightarrow -6.02<-2x<-5.98 \Rightarrow 3.01>x>2.99$ or $2.99<x<3.01$.

Step 2: $0 < |x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3.$
Then $-\delta + 3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta + 3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

32. $\lim\limits_{x\to -1}(-3x-2) = (-3)(-1) - 2 = 1$
Step 1: $|(-3x-2)-1| < 0.03 \Rightarrow -0.03 < -3x-3 < 0.03 \Rightarrow 0.01 > x+1 > -0.01 \Rightarrow -1.01 < x < -0.99.$
Step 2: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1.$
Then $-\delta - 1 = -1.01 \Rightarrow \delta = 0.01$, or $\delta - 1 = -0.99 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

33. $\lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \lim\limits_{x\to 2}\frac{(x+2)(x-2)}{(x-2)} = \lim\limits_{x\to 2}(x+2) = 2+2 = 4,\ x \ne 2$
Step 1: $\left|\left(\frac{x^2-4}{x-2}\right) - 4\right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)} - 4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05,\ x \ne 2$
$\Rightarrow 1.95 < x < 2.05,\ x \ne 2.$
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2.$
Then $-\delta + 2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta + 2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

34. $\lim\limits_{x\to -5}\frac{x^2+6x+5}{x+5} = \lim\limits_{x\to -5}\frac{(x+5)(x+1)}{(x+5)} = \lim\limits_{x\to -5}(x+1) = -4,\ x \ne -5.$
Step 1: $\left|\left(\frac{x^2+6x+5}{x+5}\right) - (-4)\right| < 0.05 \Rightarrow -0.05 < \frac{(x+5)(x+1)}{(x+5)} + 4 < 0.05 \Rightarrow -4.05 < x+1 < -3.95,\ x \ne -5$
$\Rightarrow -5.05 < x < -4.95,\ x \ne -5.$
Step 2: $|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta - 5 < x < \delta - 5.$
Then $-\delta - 5 = -5.05 \Rightarrow \delta = 0.05$, or $\delta - 5 = -4.95 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

35. $\lim\limits_{x\to -3}\sqrt{1-5x} = \sqrt{1-5(-3)} = \sqrt{16} = 4$
Step 1: $\left|\sqrt{1-5x} - 4\right| < 0.5 \Rightarrow -0.5 < \sqrt{1-5x} - 4 < 0.5 \Rightarrow 3.5 < \sqrt{1-5x} < 4.5 \Rightarrow 12.25 < 1-5x < 20.25$
$\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < 2.25.$
Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3.$
Then $-\delta - 3 = -3.85 \Rightarrow \delta = 0.85$, or $\delta - 3 = -2.25 \Rightarrow 0.75$; thus $\delta = 0.75$.

36. $\lim\limits_{x\to 2}\frac{4}{x} = \frac{4}{2} = 2$
Step 1: $\left|\frac{4}{x} - 2\right| < 0.4 \Rightarrow -0.4 < \frac{4}{x} - 2 < 0.4 \Rightarrow 1.6 < \frac{4}{x} < 2.4 \Rightarrow \frac{10}{16} > \frac{x}{4} > \frac{10}{24} \Rightarrow \frac{10}{4} > x > \frac{10}{6}$ or $\frac{5}{3} < x < \frac{5}{2}$
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2.$
Then $-\delta + 2 = \frac{5}{3} \Rightarrow \delta = \frac{1}{3}$, or $\delta + 2 = \frac{5}{2} \Rightarrow \delta = \frac{1}{2}$; thus $\delta = \frac{1}{3}$

37. Step 1: $|(9-x)-5| < \epsilon \Rightarrow -\epsilon < 4-x < \epsilon \Rightarrow -\epsilon - 4 < -x < \epsilon - 4 \Rightarrow \epsilon + 4 > x > 4 - \epsilon \Rightarrow 4 - \epsilon < x < 4 + \epsilon.$
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4.$
Then $-\delta + 4 = -\epsilon + 4 \Rightarrow \delta = \epsilon$, or $\delta + 4 = \epsilon + 4 \Rightarrow \delta = \epsilon$. Thus choose $\delta = \epsilon$.

38. Step 1: $|(3x-7)-2| < \epsilon \Rightarrow -\epsilon < 3x-9 < \epsilon \Rightarrow 9 - \epsilon < 3x < 9 + \epsilon \Rightarrow 3 - \frac{\epsilon}{3} < x < 3 + \frac{\epsilon}{3}.$
Step 2: $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3.$
Then $-\delta + 3 = 3 - \frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$, or $\delta + 3 = 3 + \frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$. Thus choose $\delta = \frac{\epsilon}{3}$.

39. Step 1: $\left|\sqrt{x-5} - 2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{x-5} - 2 < \epsilon \Rightarrow 2 - \epsilon < \sqrt{x-5} < 2 + \epsilon \Rightarrow (2-\epsilon)^2 < x-5 < (2+\epsilon)^2$
$\Rightarrow (2-\epsilon)^2 + 5 < x < (2+\epsilon)^2 + 5.$
Step 2: $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta + 9 < x < \delta + 9.$

Then $-\delta + 9 = \square^2 - 4\square + 9 \Rightarrow \delta = 4\square - \square^2$, or $\delta + 9 = \square^2 + 4\square + 9 \Rightarrow \delta = 4\square + \square^2$. Thus choose the smaller distance, $\delta = 4\square - \square^2$.

40. Step 1: $\left|\sqrt{4-x}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{4-x}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{4-x}<2+\epsilon \Rightarrow (2-\epsilon)^2<4-x<(2+\epsilon)^2$
$\Rightarrow -(2+\epsilon)^2<x-4<-(2-\epsilon)^2 \Rightarrow -(2+\epsilon)^2+4<x<-(2-\epsilon)^2+4.$

Step 2: $|x-0|<\delta \Rightarrow -\delta<x<\delta.$
Then $-\delta=-(2+\epsilon)^2+4=-\epsilon^2-4\epsilon \Rightarrow \delta=4\epsilon+\epsilon^2$, or $\delta=-(2-\epsilon)^2+4=4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

41. Step 1: For $x\neq 1$, $\left|x^2-1\right|<\epsilon \Rightarrow -\epsilon<x^2-1<\epsilon \Rightarrow 1-\epsilon<x^2<1+\epsilon \Rightarrow \sqrt{1-\epsilon}<|x|<\sqrt{1+\epsilon}$
$\Rightarrow \sqrt{1-\epsilon}<x<\sqrt{1+\epsilon}$ near $x=1$.

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1.$
Then $-\delta+1=\sqrt{1-\epsilon} \Rightarrow \delta=1-\sqrt{1-\epsilon}$, or $\delta+1=\sqrt{1+\epsilon} \Rightarrow \delta=\sqrt{1+\epsilon}-1$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, that is, the smaller of the two distances.

42. Step 1: For $x\neq -2$, $\left|x^2-4\right|<\epsilon \Rightarrow -\epsilon<x^2-4<\epsilon \Rightarrow 4-\epsilon<x^2<4+\epsilon \Rightarrow \sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon} \Rightarrow -\sqrt{4+\epsilon}<x$
$<-\sqrt{4-\epsilon}$ near $x=-2$.

Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2.$
Then $-\delta-2=-\sqrt{4+\epsilon} \Rightarrow \delta=\sqrt{4+\epsilon}-2$, or $\delta-2=-\sqrt{4-\epsilon} \Rightarrow \delta=2-\sqrt{4-\epsilon}$. Choose
$\delta=\min\left\{\sqrt{4+\epsilon}-2, 2-\sqrt{4-\epsilon}\right\}.$

43. Step 1: $\left|\frac{1}{x}-1\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x}-1<\epsilon \Rightarrow 1-\epsilon<\frac{1}{x}<1+\epsilon \Rightarrow \frac{1}{1+\epsilon}<x<\frac{1}{1-\epsilon}.$

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=\frac{1}{1+\epsilon} \Rightarrow \delta=1-\frac{1}{1+\epsilon}=\frac{\epsilon}{1+\epsilon}$, or $1+\delta=\frac{1}{1-\epsilon} \Rightarrow \delta=\frac{1}{1-\epsilon}-1=\frac{\epsilon}{1-\epsilon}.$
Choose $\delta=\frac{\epsilon}{1+\epsilon}$, the smaller of the two distances.

44. Step 1: $\left|\frac{1}{x^2}-\frac{1}{3}\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x^2}-\frac{1}{3}<\epsilon \Rightarrow \frac{1}{3}-\epsilon<\frac{1}{x^2}<\frac{1}{3}+\epsilon \Rightarrow \frac{1-3\epsilon}{3}<\frac{1}{x^2}<\frac{1+3\epsilon}{3}$
$\Rightarrow \frac{3}{1-3\epsilon}>x^2>\frac{3}{1+3\epsilon} \Rightarrow \sqrt{\frac{3}{1+3\epsilon}}<|x|<\sqrt{\frac{3}{1-3\epsilon}}$, or $\sqrt{\frac{3}{1+3\epsilon}}<x<\sqrt{\frac{3}{1-3\epsilon}}$ for x near $\sqrt{3}$.

Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow \sqrt{3}-\delta<x<\sqrt{3}+\delta.$
Then $\sqrt{3}-\delta=\sqrt{\frac{3}{1+3\epsilon}} \Rightarrow \delta=\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}$, or $\sqrt{3}+\delta=\sqrt{\frac{3}{1-3\epsilon}} \Rightarrow \delta=\sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}.$
Choose $\delta=\min\left\{\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}, \sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}\right\}.$

45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right|<\epsilon \Rightarrow -\epsilon<(x-3)+6<\epsilon, x\neq -3 \Rightarrow -\epsilon<x+3<\epsilon \Rightarrow -\epsilon-3<x<\epsilon-3.$

Step 2: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3.$
Then $-\delta-3=-\epsilon-3 \Rightarrow \delta=\epsilon$, or $\delta-3=\epsilon-3 \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

46. Step 1: $\left|\left(\frac{x^2-1}{x-1}\right)-2\right|<\epsilon \Rightarrow -\epsilon<(x+1)-2<\epsilon, x\neq 1 \Rightarrow 1-\epsilon<x<1+\epsilon.$

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\epsilon \Rightarrow \delta=\epsilon$, or $1+\delta=1+\epsilon \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

47. Step 1: $x<1$: $|(4-2x)-2|<\epsilon \Rightarrow 0<2-2x<\epsilon$ since $x<1$. Thus, $1-\frac{\epsilon}{2}<x<0$;
$x\geq 1$: $|(6x-4)-2|<\epsilon \Rightarrow 0\leq 6x-6<\epsilon$ since $x\geq 1$. Thus, $1\leq x<1+\frac{\epsilon}{6}$.

Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\frac{\epsilon}{2} \Rightarrow \delta=\frac{\epsilon}{2}$, or $1+\delta=1+\frac{\epsilon}{6} \Rightarrow \delta=\frac{\epsilon}{6}$. Choose $\delta=\frac{\epsilon}{6}$.

48. Step 1: $x < 0: |2x - 0| < \epsilon \Rightarrow -\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0;$
$x \ge 0: \left|\frac{x}{2} - 0\right| < \epsilon \Rightarrow 0 \le x < 2\epsilon.$
Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta.$
Then $-\delta = -\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $\delta = 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = \frac{\epsilon}{2}$.

49. By the figure, $-x \le x \sin\frac{1}{x} \le x$ for all $x > 0$ and $-x \ge x \sin\frac{1}{x} \ge x$ for $x < 0$. Since $\lim\limits_{x\to 0}(-x) = \lim\limits_{x\to 0} x = 0$, then by the sandwich theorem, in either case, $\lim\limits_{x\to 0} x \sin\frac{1}{x} = 0$.

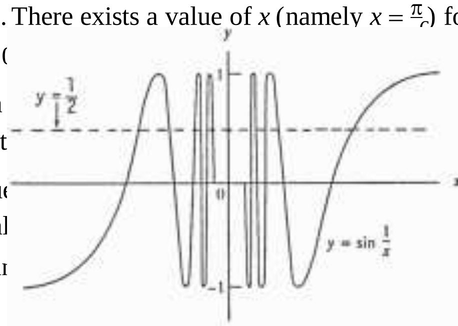
50. By the figure, $-x^2 \le x^2 \sin\frac{1}{x} \le x^2$ for all x except possibly at $x = 0$. Since $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$, then by the sandwich theorem, $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

51. As x approaches the value 0, the values of $g(x)$ approach k. Thus for every number $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |g(x) - k| < \epsilon$.

52. Write $x = h + c$. Then $0 < |x - c| < \delta \Leftrightarrow -\delta < x - c < \delta,\ x \ne c \Leftrightarrow -\delta < (h + c) - c < \delta,\ h + c \ne c \Leftrightarrow -\delta < h < \delta,$ $h \ne 0 \Leftrightarrow 0 < |h - 0| < \delta.$
Thus, $\lim\limits_{x\to c} f(x) = L \Leftrightarrow$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta \Leftrightarrow$ $|f(h + c) - L| < \epsilon$ whenever $0 < |h - 0| < \delta \Leftrightarrow \lim\limits_{h\to 0} f(h + c) = L.$

53. Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim\limits_{x\to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily</u> <u>close</u> to -1 for x near 0.

54. Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely $x = \frac{\pi}{6}$) for which $\left|\sin x - \frac{1}{2}\right| < \epsilon$ for any given $\epsilon > 0$. However, $\lim\limits_{x\to 0} \sin x = ($ … $)$ … ire x to be arbitrarily close to x_0. As another example, let $g(x) = \sin$ … ely many values of x near 0 such that $\sin\frac{1}{x} = \frac{1}{2}$ as you can see from t … $\frac{1}{x}$ fails to exist. The wrong statement does not require <u>all</u> valu … in $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are al … hat $\sin\frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$ we cannot satisfy the inequality $|\sin$ … ar $x_0 = 0$.



55. $|A - 9| \le 0.01 \Rightarrow -0.01 \le \pi\left(\frac{x}{2}\right)^2 - 9 \le 0.01 \Rightarrow 8.99 \le \frac{\pi x^2}{4} \le 9.01 \Rightarrow \frac{4}{\pi}(8.99) \le x^2 \le \frac{4}{\pi}(9.01)$
$\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \le x \le 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \le x \le 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

56. $V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left|\frac{V}{R} - 5\right| \le 0.1 \Rightarrow -0.1 \le \frac{120}{R} - 5 \le 0.1 \Rightarrow 4.9 \le \frac{120}{R} \le 5.1 \Rightarrow \frac{10}{49} \ge \frac{R}{120} \ge \frac{10}{51}$
$\Rightarrow \frac{(120)(10)}{51} \le R \le \frac{(120)(10)}{49} \Rightarrow 23.53 \le R \le 24.48.$

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x.$ Then $|f(x)-2| = |x-2| = 2-x > 2-1 = 1.$ That is, $|f(x)-2| \ge 1 \ge \frac{1}{2}$ no matter how small δ is taken when $1-\delta < x < 1 \Rightarrow \lim_{x\to 1} f(x) \ne 2.$

(b) $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1.$ Then $|f(x)-1| = |(x+1)-1| = |x| = x > 1.$ That is, $|f(x)-1| \ge 1$ no matter how small δ is taken when $1 < x < 1+\delta \Rightarrow \lim_{x\to 1} f(x) \ne 1.$

(c) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x.$ Then $|f(x)-1.5| = |x-1.5| = 1.5-x > 1.5-1 = 0.5.$
Also, $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1.$ Then $|f(x)-1.5| = |(x+1)-1.5| = |x-0.5|$ $= x-0.5 > 1-0.5 = 0.5.$ Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x-1 < \delta$ but $|f(x)-1.5| \ge \frac{1}{2} \Rightarrow \lim_{x\to 1} f(x) \ne 1.5.$

58. (a) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-4| = 2.$ Thus for □< 2, $|h(x)-4| \ge$ □whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 4.$

(b) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-3| = 1.$ Thus for □< 1, $|h(x)-3| \ge$ □whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 3.$

(c) For $2-\delta < x < 2 \Rightarrow h(x) = x^2$ so $|h(x)-2| = |x^2-2|.$ No matter how small $\delta > 0$ is chosen, x^2 is close to 4 when x is near 2 and to the left on the real line $\Rightarrow |x^2-2|$ will be close to 2. Thus if □< 1, $|h(x)-2| \ge$ □ whenever $2-\delta < x < 2$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 2.$

59. (a) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-4| \ge 0.8.$ Thus for □< 0.8, $|f(x)-4| \ge$ □whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 4.$

(b) For $3 < x < 3+\delta \Rightarrow f(x) < 3 \Rightarrow |f(x)-4.8| \ge 1.8.$ Thus for □< 1.8, $|f(x)-4.8| \ge$ □whenever $3 < x < 3+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 4.8.$

(c) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-3| \ge 1.8.$ Again, for □< 1.8, $|f(x)-3| \ge$ □whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 3.$

60. (a) No matter how small we choose $\delta > 0$, for x near -1 satisfying $-1-\delta < x < -1+\delta$, the values of $g(x)$ are near $1 \Rightarrow |g(x)-2|$ is near 1. Then, for □$= \frac{1}{2}$ we have $|g(x)-2| \ge \frac{1}{2}$ for some x satisfying $-1-\delta < x < -1+\delta$, or $0 < |x+1| < \delta \Rightarrow \lim_{x\to -1} g(x) \ne 2.$

(b) Yes, $\lim_{x\to -1} g(x) = 1$ because from the graph we can find a $\delta > 0$ such that $|g(x)-1| <$ □if $0 < |x-(-1)| < \delta.$

61–66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3); x0 := 3;
plot( f(x), x=x0-1..x0+1, color=black,                # (a)
      title="Section 2.3, #61(a)" );
L := limit( f(x), x=x0 );                             # (b)
epsilon := 0.2;                                       # (c)
plot( [f(x), L-epsilon, L+epsilon], x=x0-0.01..x0+0.01,
      color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)" );
```

```
q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1 );          # (d)
delta := abs(x0-q);
plot( [f(x), L-epsilon, L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)" );
for eps in [0.1, 0.005, 0.001 ] do                              # (e)
q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1 );
delta := abs(x0-q);
head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f \n", eps, delta );
print(plot( [f(x), L-eps, L+eps], x=x0-delta..x0+delta,
color=black, linestyle=[1,3,3], title=head ));
end do:
```

Mathematica (assigned function and values for x0, eps and del may vary):

```
Clear[f, x]
y1: = L − eps; y2: = L + eps; x0 = 1;
f[x_]: = (3x^2 − (7x + 1)Sqrt[x] + 5)/(x − 1)
Plot[f[x], {x, x0 − 0.2, x0 + 0.2}]
L: = Limit[f[x], x → x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2}, {x, x0 − del, x0 + del}, PlotRange → {L − 2eps, L + 2eps}]
```

2.4 ONE-SIDED LIMITS

1. (a) True (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

2. (a) True (b) False (c) False (d) True
(e) True (f) True (g) True (h) True
(i) True (j) False (k) True

3. (a) $\lim_{x\to 2^+} f(x) = \frac{2}{2}+1 = 2,\ \lim_{x\to 2^-} f(x) = 3-2 = 1$
(b) No, $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$
(c) $\lim_{x\to 4^-} f(x) = \frac{4}{2}+1 = 3,\ \lim_{x\to 4^+} f(x) = \frac{4}{2}+1 = 3$
(d) Yes, $\lim_{x\to 4} f(x) = 3$ because $3 = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$

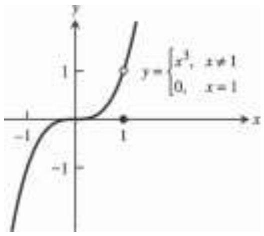
4. (a) $\lim_{x\to 2^+} f(x) = \frac{2}{2} = 1,\ \lim_{x\to 2^-} f(x) = 3-2 = 1,\ f(2) = 2$
(b) Yes, $\lim_{x\to 2} f(x) = 1$ because $1 = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$
(c) $\lim_{x\to -1^-} f(x) = 3-(-1) = 4,\ \lim_{x\to -1^+} f(x) = 3-(-1) = 4$
(d) Yes, $\lim_{x\to -1} f(x) = 4$ because $4 = \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

5. (a) No, $\lim_{x\to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0
(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$
(c) $\lim_{x\to 0} f(x)$ does not exist because

$\lim_{x \to 0^+} f(x)$ does not exist

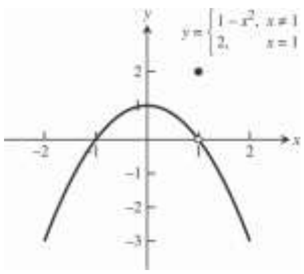
6. (a) Yes, $\lim_{x\to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \le g(x) \le \sqrt{x}$ when $x > 0$
   (b) No, $\lim_{x\to 0^-} g(x)$ does not exist since $\sqrt{x}$ is not defined for $x < 0$
   (c) Yes, $\lim_{x\to 0} g(x) = \lim_{x\to 0^+} g(x) = 0$ since $x = 0$ is a boundary point of the domain

7. (a)



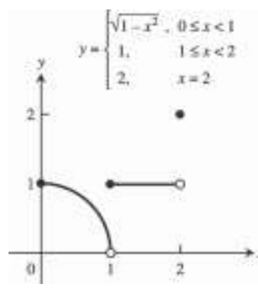
   (b) $\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x)$
   (c) Yes, $\lim_{x\to 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

8. (a)

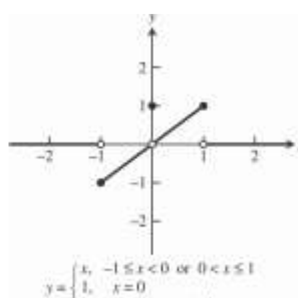


   (b) $\lim_{x\to 1^+} f(x) = 0 = \lim_{x\to 1^-} f(x)$
   (c) Yes, $\lim_{x\to 1} f(x) = 0$ since the right-hand and left-hand limits exist and equal 0

9. (a) domain: $0 \le x \le 2$
   range: $0 < y \le 1$ and $y = 2$
   (b) $\lim_{x\to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
   (c) $x = 2$
   (d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
    range: $-1 \le y \le 1$
    (b) $\lim_{x\to c} f(x)$ exists for c belonging to $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
    (c) none
    (d) none



11. $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

12. $\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}} = \sqrt{0} = 0$

13. $\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

14. $\lim\limits_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right) = \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = 1$

15. $\lim\limits_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} = \lim\limits_{h\to 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right) = \lim\limits_{h\to 0^+} \frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}$

$= \lim\limits_{h\to 0^+} \frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$

16. $\lim\limits_{h\to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim\limits_{h\to 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$= \lim\limits_{h\to 0^-} \frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \lim\limits_{h\to 0^-} \frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}}$

17. (a) $\lim\limits_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x\to -2^+} (x+3)\frac{(x+2)}{(x+2)}$ $\quad (|x+2| = (x+2) \text{ for } x > -2)$

$= \lim\limits_{x\to -2^+} (x+3) = ((-2)+3) = 1$

(b) $\lim\limits_{x\to -2^-} (x+3)\frac{|x+2|}{x+2} = \lim\limits_{x\to -2^-} (x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad (|x+2| = -(x+2) \text{ for } x < -2)$

$= \lim\limits_{x\to -2^-} (x+3)(-1) = -(-2+3) = -1$

18. (a) $\lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x\to 1^+} \frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad (|x-1| = x-1 \text{ for } x > 1)$

$= \lim\limits_{x\to 1^+} \sqrt{2x} = \sqrt{2}$

(b) $\lim\limits_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim\limits_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad (|x-1| = -(x-1) \text{ for } x < 1)$

$= \lim\limits_{x\to 1^-} -\sqrt{2x} = -\sqrt{2}$

19. (a) If $0 < x < \frac{\pi}{2}$, then $\sin x > 0$, so that $\lim\limits_{x\to 0^+} \frac{|\sin x|}{\sin x} = \lim\limits_{x\to 0^+} \frac{\sin x}{\sin x} = \lim\limits_{x\to 0^+} 1 = 1$

(b) If $\frac{-\pi}{2} < x < 0$, then $\sin x < 0$, so that $\lim\limits_{x\to 0^-} \frac{|\sin x|}{\sin x} = \lim\limits_{x\to 0^-} \frac{-\sin x}{\sin x} = \lim\limits_{x\to 0^-} -1 = -1$

20. (a) If $0 < x < \frac{\pi}{2}$, then $\cos x < 1$, so that $\lim\limits_{x\to 0^+} \frac{1-\cos x}{|\cos x - 1|} = \lim\limits_{x\to 0^+} \frac{1-\cos x}{-(\cos x - 1)} = \lim\limits_{x\to 0^+} \frac{1-\cos x}{1-\cos x} = \lim\limits_{x\to 0^+} 1 = 1$

(b) If $\frac{-\pi}{2} < x < 0$, then $\cos x < 1$, so that $\lim\limits_{x\to 0^-} \frac{\cos x - 1}{|\cos x - 1|} = \lim\limits_{x\to 0^-} \frac{\cos x - 1}{-(\cos x - 1)} = \lim\limits_{x\to 0^-} -1 = -1$

21. (a) $\lim\limits_{\theta\to 3^+} \frac{\lfloor\theta\rfloor}{\theta} = \frac{3}{3} = 1$ (b) $\lim\limits_{\theta\to 3^-} \frac{\lfloor\theta\rfloor}{\theta} = \frac{2}{3}$

22. (a) $\lim\limits_{t\to 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0$ (b) $\lim\limits_{t\to 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$

23. $\lim \frac{\sin \sqrt{2\theta}}{\sqrt{2\theta}} = \lim \frac{\sin x}{x} = 1$ $\quad$ (where $x = \sqrt{2\theta}$)

$\theta \to 0$ $\sqrt{\ }$ $\sqrt{2}\theta$ $x \to 0$ x

24. $\lim_{t\to 0} \frac{\sin kt}{t} = \lim_{t\to 0} \frac{k \sin kt}{kt} = \lim_{\theta\to 0} \frac{k \sin\theta}{\theta} = k \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = k \cdot 1 = k$ (where $\theta = kt$)

25. $\lim\limits_{y\to 0}\frac{\sin 3y}{4y}=\frac{1}{4}\lim\limits_{y\to 0}\frac{3\sin 3y}{3y}=\frac{3}{4}\lim\limits_{y\to 0}\frac{\sin 3y}{3y}=\frac{3}{4}\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=\frac{3}{4}$ (where $\theta=3y$)

26. $\lim\limits_{h\to 0^-}\frac{h}{\sin 3h}=\lim\limits_{h\to 0^-}\left(\frac{1}{3}\cdot\frac{3h}{\sin 3h}\right)=\frac{1}{3}\lim\limits_{h\to 0^-}\frac{1}{\left(\frac{\sin 3h}{3h}\right)}=\frac{1}{3}\left(\frac{1}{\lim\limits_{\theta\to 0^-}\frac{\sin\theta}{\theta}}\right)=\frac{1}{3}\cdot 1=\frac{1}{3}$ (where $\theta=3h$)

27. $\lim\limits_{x\to 0}\frac{\tan 2x}{x}=\lim\limits_{x\to 0}\frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x}=\lim\limits_{x\to 0}\frac{\sin 2x}{x\cos 2x}=\left(\lim\limits_{x\to 0}\frac{1}{\cos 2x}\right)\left(\lim\limits_{x\to 0}\frac{2\sin 2x}{2x}\right)=1\cdot 2=2$

28. $\lim\limits_{t\to 0}\frac{2t}{\tan t}=2\lim\limits_{t\to 0}\frac{t}{\left(\frac{\sin t}{\cos t}\right)}=2\lim\limits_{t\to 0}\frac{t\cos t}{\sin t}=2\left(\lim\limits_{t\to 0}\cos t\right)\left(\frac{1}{\lim\limits_{t\to 0}\frac{\sin t}{t}}\right)=2\cdot 1\cdot 1=2$

29. $\lim\limits_{x\to 0}\frac{x\csc 2x}{\cos 5x}=\lim\limits_{x\to 0}\left(\frac{x}{\sin 2x}\cdot\frac{1}{\cos 5x}\right)=\left(\frac{1}{2}\lim\limits_{x\to 0}\frac{2x}{\sin 2x}\right)\left(\lim\limits_{x\to 0}\frac{1}{\cos 5x}\right)=\left(\frac{1}{2}\cdot 1\right)(1)=\frac{1}{2}$

30. $\lim\limits_{x\to 0}6x^2(\cot x)(\csc 2x)=\lim\limits_{x\to 0}\frac{6x^2\cos x}{\sin x\sin 2x}=\lim\limits_{x\to 0}\left(3\cos x\cdot\frac{x}{\sin x}\cdot\frac{2x}{\sin 2x}\right)=3\cdot 1\cdot 1=3$

31. $\lim\limits_{x\to 0}\frac{x+x\cos x}{\sin x\cos x}=\lim\limits_{x\to 0}\left(\frac{x}{\sin x\cos x}+\frac{x\cos x}{\sin x\cos x}\right)=\lim\limits_{x\to 0}\left(\frac{x}{\sin x}\cdot\frac{1}{\cos x}\right)+\lim\limits_{x\to 0}\frac{x}{\sin x}$

$=\lim\limits_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)\cdot\lim\limits_{x\to 0}\left(\frac{1}{\cos x}\right)+\lim\limits_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right)=(1)(1)+1=2$

32. $\lim\limits_{x\to 0}\frac{x^2-x+\sin x}{2x}=\lim\limits_{x\to 0}\left(\frac{x}{2}-\frac{1}{2}+\frac{1}{2}\left(\frac{\sin x}{x}\right)\right)=0-\frac{1}{2}+\frac{1}{2}(1)=0$

33. $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\sin 2\theta}=\lim\limits_{\theta\to 0}\frac{(1-\cos\theta)(1+\cos\theta)}{(2\sin\theta\cos\theta)(1+\cos\theta)}=\lim\limits_{\theta\to 0}\frac{1-\cos^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}=\lim\limits_{\theta\to 0}\frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$

$=\lim\limits_{\theta\to 0}\frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)}=\frac{0}{(2)(2)}=0$

34. $\lim\limits_{x\to 0}\frac{x-x\cos x}{\sin^2 3x}=\lim\limits_{x\to 0}\frac{x(1-\cos x)}{\sin^2 3x}=\lim\limits_{x\to 0}\frac{\frac{x(1-\cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}}=\lim\limits_{x\to 0}\frac{\frac{1-\cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2}=\frac{\frac{1}{9}\lim\limits_{x\to 0}\left(\frac{1-\cos x}{x}\right)}{\lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2}=\frac{\frac{1}{9}(0)}{1^2}=0$

35. $\lim\limits_{t\to 0}\frac{\sin(1-\cos t)}{1-\cos t}=\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ since $\theta=1-\cos t\to 0$ as $t\to 0$

36. $\lim\limits_{h\to 0}\frac{\sin(\sin h)}{\sin h}=\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ since $\theta=\sin h\to 0$ as $h\to 0$

37. $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}=\lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\sin 2\theta}\cdot\frac{2\theta}{2\theta}\right)=\frac{1}{2}\lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{2\theta}{\sin 2\theta}\right)=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}$

38. $\lim\limits_{x\to 0}\frac{\sin 5x}{\sin 4x}=\lim\limits_{x\to 0}\left(\frac{\sin 5x}{\sin 4x}\cdot\frac{4x}{5x}\cdot\frac{5}{4}\right)=\frac{5}{4}\lim\limits_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{4x}{\sin 4x}\right)=\frac{5}{4}\cdot 1\cdot 1=\frac{5}{4}$

39. $\lim_{\theta \to 0} \theta \cos\theta = 0 \cdot 1 = 0$

40. $\lim_{\theta \to 0} \sin\theta \cot 2\theta = \lim_{\theta \to 0} \sin\theta \frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta \to 0} \sin\theta \frac{\cos 2\theta}{2\sin\theta\cos\theta} = \lim_{\theta \to 0} \frac{\cos 2\theta}{2\cos\theta} = \frac{1}{2}$

41. $\lim\limits_{x\to0}\frac{\tan3x}{\sin8x}=\lim\limits_{x\to0}\left(\frac{\sin3x}{\cos3x}\cdot\frac{1}{\sin8x}\right)=\lim\limits_{x\to0}\left(\frac{\sin3x}{\cos3x}\cdot\frac{1}{\sin8x}\cdot\frac{8x}{3x}\cdot\frac{3}{8}\right)$

$=\frac{3}{8}\lim\limits_{x\to0}\left(\frac{1}{\cos3x}\right)\left(\frac{\sin3x}{3x}\right)\left(\frac{8x}{\sin8x}\right)=\frac{3}{8}\cdot1\cdot1\cdot1=\frac{3}{8}$

42. $\lim\limits_{y\to0}\frac{\sin3y\cot5y}{y\cot4y}=\lim\limits_{y\to0}\frac{\sin3y\sin4y\cos5y}{y\cos4y\sin5y}=\lim\limits_{y\to0}\left(\frac{\sin3y}{y}\right)\left(\frac{\sin4y}{\cos4y}\right)\left(\frac{\cos5y}{\sin5y}\right)\left(\frac{3\cdot4\cdot5y}{3\cdot4\cdot5y}\right)$

$=\lim\limits_{y\to0}\left(\frac{\sin3y}{3y}\right)\left(\frac{\sin4y}{4y}\right)\left(\frac{5y}{\sin5y}\right)\left(\frac{\cos5y}{\cos4y}\right)\left(\frac{3\cdot4}{5}\right)=1\cdot1\cdot1\cdot1\cdot\frac{12}{5}=\frac{12}{5}$

43. $\lim\limits_{\theta\to0}\frac{\tan\theta}{\theta^2\cot3\theta}=\lim\limits_{\theta\to0}\frac{\frac{\sin\theta}{\cos\theta}}{\theta^2\frac{\cos3\theta}{\sin3\theta}}=\lim\limits_{\theta\to0}\frac{\sin\theta\sin3\theta}{\theta^2\cos\theta\cos3\theta}=\lim\limits_{\theta\to0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos3\theta}\right)=(1)(1)\left(\frac{3}{1\cdot1}\right)=3$

44. $\lim\limits_{\theta\to0}\frac{\theta\cot4\theta}{\sin^2\theta\cot^22\theta}=\lim\limits_{\theta\to0}\frac{\theta\frac{\cos4\theta}{\sin4\theta}}{\sin^2\theta\frac{\cos^22\theta}{\sin^22\theta}}=\lim\limits_{\theta\to0}\frac{\theta\cos4\theta\sin^22\theta}{\sin^2\theta\cos^22\theta\sin4\theta}=\lim\limits_{\theta\to0}\frac{\theta\cos4\theta(2\sin\theta\cos\theta)^2}{\sin^2\theta\cos^22\theta\sin4\theta}=\lim\limits_{\theta\to0}\frac{\theta\cos4\theta(4\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^22\theta\sin4\theta}$

$=\lim\limits_{\theta\to0}\frac{4\theta\cos4\theta\cos^2\theta}{\cos^22\theta\sin4\theta}=\lim\limits_{\theta\to0}\left(\frac{4\theta}{\sin4\theta}\right)\left(\frac{\cos4\theta\cos^2\theta}{\cos^22\theta}\right)=\lim\limits_{\theta\to0}\left(\frac{1}{\frac{\sin4\theta}{4\theta}}\right)\left(\frac{\cos4\theta\cos^2\theta}{\cos^22\theta}\right)=\left(\frac{1}{1}\right)\left(\frac{1\cdot1^2}{1^2}\right)=1$

45. $\lim\limits_{x\to0}\frac{1-\cos3x}{2x}=\lim\limits_{x\to0}\frac{1-\cos3x}{2x}\cdot\frac{1+\cos3x}{1+\cos3x}=\lim\limits_{x\to0}\frac{1-\cos^23x}{2x(1+\cos3x)}=\lim\limits_{x\to0}\frac{\sin^23x}{2x(1+\cos3x)}=\lim\limits_{x\to0}\frac{3}{2}\cdot\frac{\sin3x}{3x}\cdot\frac{\sin3x}{1+\cos3x}$

$=\lim\limits_{\theta\to0}\frac{3}{2}\cdot\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}=\frac{3}{2}(1)\left(\frac{0}{1+1}\right)=0$ (where $\theta=3x$)

46. $\lim\limits_{x\to0}\frac{\cos^2x-\cos x}{x^2}=\lim\limits_{x\to0}\frac{\cos x(\cos x-1)}{x^2}=\lim\limits_{x\to0}\frac{\cos x(\cos x-1)}{x^2}\cdot\frac{\cos x+1}{\cos x+1}=\lim\limits_{x\to0}\frac{\cos x(\cos^2x-1)}{x^2(\cos x+1)}=\lim\limits_{x\to0}\frac{\cos x\cdot(-\sin^2x)}{x^2(\cos x+1)}$

$=\lim\limits_{x\to0}\left\{-\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{\cos x}{\cos x+1}\right\}=-(1)(1)\cdot\frac{1}{1+1}=\frac{-1}{2}$

47. Yes. If $\lim\limits_{x\to a^+}f(x)=L=\lim\limits_{x\to a^-}f(x)$, then $\lim\limits_{x\to a}f(x)=L$. If $\lim\limits_{x\to a^+}f(x)\neq\lim\limits_{x\to a^-}f(x)$, then $\lim\limits_{x\to a}f(x)$ does not exist.

48. Since $\lim\limits_{x\to c}f(x)=L$ if and only if $\lim\limits_{x\to c^+}f(x)=L$ and $\lim\limits_{x\to c^-}f(x)=L$, then $\lim\limits_{x\to c}f(x)$ can be found by calculating $\lim\limits_{x\to c^+}f(x)$.

49. If f is an odd function of x, then $f(-x)=-f(x)$. Given $\lim\limits_{x\to0^+}f(x)=3$, then $\lim\limits_{x\to0^-}f(x)=-3$.

50. If f is an even function of x, then $f(-x)=f(x)$. Given $\lim\limits_{x\to2^-}f(x)=7$ then $\lim\limits_{x\to-2^+}f(x)=7$. However, nothing can be said about $\lim\limits_{x\to-2^-}f(x)$ because we don't know $\lim\limits_{x\to2^+}f(x)$.

51. $I=(5,5+\delta)\Rightarrow5<x<5+\delta$. Also, $\sqrt{x-5}<\epsilon\Rightarrow x-5<\epsilon^2\Rightarrow x<5+\epsilon^2$. Choose $\delta=\epsilon^2\Rightarrow\lim\limits_{x\to5^+}\sqrt{x-5}=0$.

52. $I=(4-\delta,4)\Rightarrow4-\delta<x<4$. Also, $\sqrt{4-x}<\epsilon\Rightarrow4-x<\epsilon^2\Rightarrow x>4-\epsilon^2$. Choose $\delta=\epsilon^2\Rightarrow\lim\limits_{x\to4^-}\sqrt{4-x}=0$.

53. As $x \to 0^-$ the number x is always negative. Thus, $\left|\frac{x}{|x|} - (-1)\right| < \square \Rightarrow \left|\frac{x}{-x} + 1\right| < \square \Rightarrow 0 < \square$ which is always true independent of the value of x. Hence we can choose any $\delta > 0$ with $-\delta < x < 0 \Rightarrow \lim\limits_{x \to 0^-} \frac{x}{|x|} = -1.$

54. Since $x \to 2^+$ we have $x > 2$ and $|x-2| = x-2$. Then, $\left|\frac{x-2}{|x-2|} - 1\right| = \left|\frac{x-2}{x-2} - 1\right| < \square \Rightarrow 0 < \square$ which is always true so long as $x > 2$. Hence we can choose any $\delta > 0$, and thus $2 < x < 2+\delta \Rightarrow \left|\frac{x-2}{|x-2|} - 1\right| < \square$ Thus, $\lim\limits_{x \to -2^+} \frac{x-2}{|x-2|} = 1.$

55. (a) $\lim\limits_{x \to 400^+} \lfloor x \rfloor = 400$. Just observe that if $400 < x < 401$, then $\lfloor x \rfloor = 400$. Thus if we choose $\delta = 1$, we have for any number $\square > 0$ that $400 < x < 400 + \delta \Rightarrow |\lfloor x \rfloor - 400| = |400 - 400| = 0 < \square$.
    (b) $\lim\limits_{x \to 400^-} \lfloor x \rfloor = 399$. Just observe that if $399 < x < 400$ then $\lfloor x \rfloor = 399$. Thus if we choose $\delta = 1$, we have for any number $\square > 0$ that $400 - \delta < x < 400 \Rightarrow |\lfloor x \rfloor - 399| = |399 - 399| = 0 < \square$.
    (c) Since $\lim\limits_{x \to 400^+} \lfloor x \rfloor \neq \lim\limits_{x \to 400^-} \lfloor x \rfloor$ we conclude that $\lim\limits_{x \to 400} \lfloor x \rfloor$ does not exist.

56. (a) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$; $\left|\sqrt{x} - 0\right| < \square \Rightarrow -\square < \sqrt{x} < \square \Rightarrow 0 < x < \square^2$ for x positive. Choose $\delta = \square^2$ $\Rightarrow \lim\limits_{x \to 0^+} f(x) = 0.$
    (b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the sandwich theorem since $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for all $x \neq 0$. Since $\left|x^2 - 0\right| = \left|-x^2 - 0\right| = x^2 < \square$ whenever $|x| < \sqrt{\square}$, we choose $\delta = \sqrt{\square}$ and obtain $\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| < \square$ if $-\delta < x < 0.$
    (c) The function f has limit 0 at $x_0 = 0$ since both the right-hand and left-hand limits exist and equal 0.

2.5 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$
2. No, discontinuous at $x = 3$, $1 = \lim\limits_{x \to 3^-} g(x) \neq g(3) = 1.5$
3. Continuous on $[-1, 3]$
4. No, discontinuous at $x = 1$, $1.5 = \lim\limits_{x \to 1^-} k(x) \neq \lim\limits_{x \to 1^+} k(x) = 0$
5. (a) Yes (b) Yes, $\lim\limits_{x \to -1^+} f(x) = 0$
    (c) Yes (d) Yes
6. (a) Yes, $f(1) = 1$ (b) Yes, $\lim\limits_{x \to 1} f(x) = 2$
    (c) No (d) No
7. (a) No (b) No
8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$
9. $f(2) = 0$, since $\lim\limits_{x \to 2^-} f(x) = -2(2) + 4 = 0 = \lim\limits_{x \to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. Nonremovable discontinuity at $x=1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x)=1$ and $\lim_{x\to 1^+} f(x)=0$). Removable discontinuity at $x=0$ by assigning the number $\lim_{x\to 0} f(x)=0$ to be the value of $f(0)$ rather than $f(0)=1$.

12. Nonremovable discontinuity at $x=1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x)=2$ and $\lim_{x\to 1^+} f(x)=1$). Removable discontinuity at $x=2$ by assigning the number $\lim_{x\to 2} f(x)=1$ to be the value of $f(2)$ rather than $f(2)=2$.

13. Discontinuous only when $x-2=0 \Rightarrow x=2$

14. Discontinuous only when $(x+2)^2=0 \Rightarrow x=-2$

15. Discontinuous only when $x^2-4x+3=0 \Rightarrow (x-3)(x-1)=0 \Rightarrow x=3$ or $x=1$

16. Discontinuous only when $x^2-3x-10=0 \Rightarrow (x-5)(x+2)=0 \Rightarrow x=5$ or $x=-2$

17. Continuous everywhere. ($|x-1|+\sin x$ defined for all x; limits exist and are equal to function values.)

18. Continuous everywhere. ($|x|+1\neq 0$ for all x; limits exist and are equal to function values.)

19. Discontinuous only at $x=0$

20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x=(2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

21. Discontinuous when $2x$ is an integer multiple of π, i.e., $2x=n\pi$, n an integer $\Rightarrow x=\frac{n\pi}{2}$, n an integer, but continuous at all other x.

22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2}=(2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow x=2n-1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x=(2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

24. Continuous everywhere since $-1\le \sin x\le 1 \Rightarrow 0\le \sin^2 x\le 1 \Rightarrow 1+\sin^2 x\ge 1$; limits exist and are equal to the function values.

25. Discontinuous when $2x+3<0$ or $x<-\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2},\infty\right)$.

26. Discontinuous when $3x-1<0$ or $x<\frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3},\infty\right)$.

27. Continuous everywhere: $(2x-1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

28. Continuous everywhere: $(2-x)^{1/5}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} (x+2) = 5 = g(3)$

30. Discontinuous at $x=-2$ since $\lim_{x\to -2} f(x)$ does not exist while $f(-2)=4$.

31. Discontinuous at $x=1$; $\lim_{x\to 1^+}(x^2+2)=3$, but $\lim_{x\to 1^-} e^x = e$, so that $\lim_{x\to 1} f(x)$ does not exist while $f(1)=e$; and $\lim_{x\to 0^-}(1-x)=1=\lim_{x\to 0^+} e^x$, so that $\lim_{x\to 0} f(x)=1=f(0)$

32. Discontinuous at $x=\ln 2$, since $2-e^x=0 \Rightarrow e^x=2 \Rightarrow \ln e^x=\ln 2 \Rightarrow x=\ln 2$

33. $\lim_{x\to\pi}\sin(x-\sin x)=\sin(\pi-\sin\pi)=\sin(\pi-0)=\sin\pi=0$, and function continuous at $x=\pi$

34. $\lim_{t\to 0}\sin(\frac{\pi}{2}\cos(\tan t))=\sin(\frac{\pi}{2}\cos(\tan(0)))=\sin\left(\frac{\pi}{2}\cos(0)\right)=\sin\left(\frac{\pi}{2}\right)=1$, and function continuous at $t=0$.

35. $\lim_{y\to 1}\sec(y\sec^2 y-\tan^2 y-1)=\lim_{y\to 1}\sec(y\sec^2 y-\sec^2 y)=\lim_{y\to 1}\sec((y-1)\sec^2 y)=\sec((1-1)\sec^2 1)=\sec 0=1$, and function continuous at $y=1$.

36. $\lim_{x\to 0}\tan\left[\frac{\pi}{4}\cos(\sin x^{1/3})\right]=\tan\left[\frac{\pi}{4}\cos(\sin(0))\right]=\tan\left(\frac{\pi}{4}\cos(0)\right)=\tan\left(\frac{\pi}{4}\right)=1$, and function continuous at $x=0$.

37. $\lim_{t\to 0}\cos\left[\frac{\pi}{\sqrt{19-3\sec 2t}}\right]=\cos\left[\frac{\pi}{\sqrt{19-3\sec 0}}\right]=\cos\frac{\pi}{\sqrt{16}}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, and function continuous at $t=0$.

38. $\lim_{x\to\frac{\pi}{6}}\sqrt{\csc^2 x+5\sqrt{3}\tan x}=\sqrt{\csc^2\left(\frac{\pi}{6}\right)+5\sqrt{3}\tan\left(\frac{\pi}{6}\right)}=\sqrt{4+5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)}=\sqrt{9}=3$, and function continuous at $x=\frac{\pi}{6}$.

39. $\lim_{x\to 0^+}\sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)=\sin\left(\frac{\pi}{2}e^0\right)=\sin\left(\frac{\pi}{2}\right)=1$, and the function is continuous at $x=0$.

40. $\lim_{x\to 1}\cos^{-1}\left(\ln\sqrt{x}\right)=\cos^{-1}\left(\ln\sqrt{1}\right)=\cos^{-1}(0)=\frac{\pi}{2}$, and the function is continuous at $x=1$.

41. $g(x)=\frac{x^2-9}{x-3}=\frac{(x+3)(x-3)}{(x-3)}=x+3,\ x\neq 3 \Rightarrow g(3)=\lim_{x\to 3}(x+3)=6$

42. $h(t)=\frac{t^2+3t-10}{t-2}=\frac{(t+5)(t-2)}{t-2}=t+5,\ t\neq 2 \Rightarrow h(2)=\lim_{t\to 2}(t+5)=7$

43. $f(s)=\frac{s^3-1}{s^3-1}=\frac{(s^2+s+1)(s-1)}{(s+1)(s-1)}=\frac{s^2+s+1}{s+1},\ s\neq 1 \Rightarrow f(1)=\lim_{s\to 1}\left(\frac{s^2+s+1}{s+1}\right)=\frac{3}{2}$

44. $g(x)=\frac{x^2-16}{x^2-3x-4}=\frac{(x+4)(x-4)}{(x-4)(x+1)}=\frac{x+4}{x+1},\ x\neq 4 \Rightarrow g(4)=\lim_{x\to 4}\left(\frac{x+4}{x+1}\right)=\frac{8}{5}$

45. As defined, $\lim_{x\to 3^-} f(x)=(3)^2-1=8$ and $\lim_{x\to 3^+}(2a)(3)=6a$. For $f(x)$ to be continuous we must have $6a=8 \Rightarrow a=\frac{4}{3}$.